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Bipolar Anti L-Fuzzy Sub ℓ-HX Group and its Lower Level Sub ℓ-HX Groups

***R. Muthuraj*[1] and *G. Santha Meena*[2]**

[1] PG & Research Department of Mathematics, H.H. The Rajah's College
Pudukkottai – 622 001, Tamilnadu, India. rmr1973@yahoo.co.in
[2]Department of Mathematics, PSNA College of Engineering and Technology
Dindigul – 624 622, Tamilnadu, India. g.santhameena@gmail.com

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Abstract. In this paper, we discussed some properties of bipolar anti L - fuzzy sub ℓ - HX group of aℓ - HX group. The purpose of this study is to implement the fuzzy set theory and graph theory in bipolar anti L - fuzzy sub ℓ - HX group. Characterizations of lower level subsets of a bipolar anti L - fuzzy sub ℓ - HX group are given. We also discussed the relation between a bipolar anti L - fuzzy sub ℓ - HX group and its lower level subℓ - HX groups and investigate the conditions under which a given sub ℓ - HX group has a properly inclusive chain of sub ℓ - HX groups. In particular, we formulate how to structure a bipolar anti L - fuzzy sub ℓ - HX group by a given chain of sub ℓ - HX groups. We also establish the relation between bipolar L - fuzzy sub ℓ - HX group and bipolar anti L - fuzzy sub ℓ - HX group.

Keywords: Bipolar L - fuzzy ℓ - HX group, Bipolar anti L - fuzzy ℓ - HX group, lower level subset, lower level sub ℓ-HX group.

AMS Mathematics Subject Classification (2010): 20N25, 03E72, 03G25

1. Introduction

The concept of fuzzy sets was initiated by Zadeh [14]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [11] gave the idea of fuzzy subgroups. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Hongxing [4] introduced the concept of HX group and the authors Chengzhong, Honghai, Hongxing [5] introduced the concept of fuzzy HX group. The author Zhang [15] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. Lee [3] introduced Bipolar-valued fuzzy sets and their operations. In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [−1,0) indicates that elements somewhat satisfy the implicit counter-property. Satya Saibaba [12] initiated the study of L - fuzzy lattice ordered groups and introduced the notions of L - fuzzy sub ℓ - HX group. Goguen [2] replaced the valuation set [0,1] by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L - Fuzzy sets. Muthuraj and Sridharan [9] introduced Bipolar Anti Fuzzy HX Group and its Lower Level sub HX groups. Sunderrajan and Senthilkumar [13] introduced Anti L-fuzzy sub ℓ group and its lower level sub ℓ groups.

2. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G , .) is a group, e is the identity element of G, and xy, we mean x .y.

Definition 2.1. A Bipolar L - fuzzy set μ in G is a bipolar L - fuzzy subgroup of G if for all $x,y \in G$.

i) $\mu^+(xy) \geq \mu^+(x) \wedge \mu^+(y)$
ii) $\mu^-(xy) \leq \mu^-(x) \vee \mu^-(y)$
iii) $\mu^+(x^{-1}) = \mu^+(x)$
iv) $\mu^-(x^{-1}) = \mu^-(x)$.

Definition 2.2. A bipolar anti L - fuzzy set μ in G is a bipolar anti L - fuzzy subgroup of G if for all $x,y \in G$.

i) $\mu^+(xy) \leq \mu^+(x) \vee \mu^+(y)$
ii) $\mu^-(xy) \geq \mu^-(x) \wedge \mu^-(y)$
iii) $\mu^+(x^{-1}) = \mu^+(x)$
iv) $\mu^-(x^{-1}) = \mu^-(x)$.

Definition 2.3. A bipolar L-fuzzy subset μ of G is said to be bipolar L-fuzzy sub ℓ group of G if for any $x,y \in G$.

i) $\mu^+(xy) \geq \mu^+(x) \wedge \mu^+(y)$
ii) $\mu^-(xy) \leq \mu^-(x) \vee \mu^-(y)$
iii) $\mu^+(x^{-1}) = \mu^+(x)$
iv) $\mu^-(x^{-1}) = \mu^-(x)$
v) $\mu^+(x \vee y) \geq \mu^+(x) \wedge \mu^+(y)$
vi) $\mu^-(x \vee y) \leq \mu^-(x) \vee \mu^-(y)$
vii) $\mu^+(x \wedge y) \geq \mu^+(x) \wedge \mu^+(y)$
viii) $\mu^-(x \wedge y) \leq \mu^-(x) \vee \mu^-(y)$

Definition 2.4. A bipolar anti L-fuzzy subset μ of G is said to be bipolar anti L-fuzzy sub ℓ group of G if for any $x,y \in G$

i) $\mu^+(xy) \leq \mu^+(x) \vee \mu^+(y)$
ii) $\mu^-(xy) \geq \mu^-(x) \wedge \mu^-(y)$

iii) $\mu^+ (x^{-1}) = \mu^+ (x)$
iv) $\mu^- (x^{-1}) = \mu^- (x)$
v) $\mu^+ (x\vee y) \le \mu^+ (x) \vee \mu^+ (y)$
vi) $\mu^- (x\vee y) \ge \mu^- (x) \wedge \mu^- (y)$
vii) $\mu^+ (x\wedge y) \le \mu^+ (x) \vee \mu^+ (y)$
viii) $\mu^- (x\wedge y) \ge \mu^- (x) \wedge \mu^- (y)$

Definition 2.5. Let μ be a bipolar L - fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a ℓ-HX group on G. A bipolar L - fuzzy set λ^μ defined on ϑ is said to be a bipolar L - fuzzy sub ℓ - HX group on ϑ if for all A,B$\in\vartheta$.

i) $(\lambda^\mu)^+ (AB) \ge (\lambda^\mu)^+ (A) \wedge (\lambda^\mu)^+ (B)$
ii) $(\lambda^\mu)^- (AB) \le (\lambda^\mu)^- (A) \vee (\lambda^\mu)^- (B)$
iii) $(\lambda^\mu)^+ (A) = (\lambda^\mu)^+ (A^{-1})$
iv) $(\lambda^\mu)^- (A) = (\lambda^\mu)^- (A^{-1})$
v) $(\lambda^\mu)^+ (A\vee B) \ge (\lambda^\mu)^+ (A) \wedge (\lambda^\mu)^+ (B)$
vi) $(\lambda^\mu)^- (A\vee B) \le (\lambda^\mu)^- (A) \vee (\lambda^\mu)^- (B)$
vii) $(\lambda^\mu)^+ (A\wedge B) \ge (\lambda^\mu)^+ (A) \wedge (\lambda^\mu)^+ (B)$
viii) $(\lambda^\mu)^- (A\wedge B) \le (\lambda^\mu)^- (A) \vee (\lambda^\mu)^- (B)$

where $(\lambda^\mu)^+ (A) = \max\{ \mu^+ (x) \,/\, \text{for all } x\in A \subseteq G \}$
and
$(\lambda^\mu)^- (A) = \min\{ \mu^- (x) \,/\, \text{for all } x\in A\subseteq G \}$

Example 2.1. Let G={Z_{10}-{0}, $\cdot_{10}$} be a group and define a bipolar L- fuzzy set μ on G as $\mu^+(1)=0.7$, $\mu^+(3)=0.4$, $\mu^+(7)=0.4$, $\mu^+(9)=0.3$ and $\mu^-(1)= - 0.8$, $\mu^-(3)= - 0.3$, $\mu^-(7)= - 0.3$, $\mu^-(9)= - 0.2$.

By routine computations, it is easy to see that μ is a bipolar L-fuzzy sub group of G.

Let ϑ = {{3, 7}, {1, 9}} be a ℓ - HX group of G.

Let us consider A = {3, 7}, B = {1, 9}.

$\cdot_{10}$	A	B
A	B	A
B	A	B

$\wedge$	A	B
A	A	B
B	B	B

$\vee$	A	B
A	A	A
B	A	B

Define $(\lambda^\mu)^+ (A) = \max\{\mu^+ (x) \,/\, \text{for all } x\in A \subseteq G \}$

And

$(\lambda^\mu)^- (A) = \min\{\mu^- (x) \,/\, \text{for all } x\in A\subseteq G \}$

Now $(\lambda^\mu)^+(A)= (\lambda^\mu)^+(\{3,7\}) = \max\{\mu^+(3), \mu^+ (7)\} = \max\{0.4,0.4\} = 0.4$

$(\lambda^\mu)^+(B)= (\lambda^\mu)^+(\{1,9\}) = \max\{\mu^+(1), \mu^+ (9)\} = \max\{0.7,0.3\} = 0.7$

$(\lambda^\mu)^+(AB) = (\lambda^\mu)^+(A) = 0.4$

$(\lambda^{\mu})^{+}(A\wedge B) = (\lambda^{\mu})^{+}(B) = 0.7$

$(\lambda^{\mu})^{+}(A\vee B) = (\lambda^{\mu})^{+}(A) = 0.4$

$(\lambda^{\mu})^{-}(A)= (\lambda^{\mu})^{-}(\{3,7\}) = \min\{\mu^{-}(3), \mu^{-}(7)\} = \min\{- 0.3, - 0.3\} = - 0.3$

$(\lambda^{\mu})^{-}(B)= (\lambda^{\mu})^{-}(\{1,9\}) = \min\{\mu^{-}(1), \mu^{-}(9)\} = \min\{ -0.8, - 0.2\} = - 0.8$

$(\lambda^{\mu})^{-}(AB)= (\lambda^{\mu})^{-}(A) = - 0.3$

$(\lambda^{\mu})^{-}(A\wedge B)= (\lambda^{\mu})^{-}(B) = - 0.8$

$(\lambda^{\mu})^{-}(A\vee B)= (\lambda^{\mu})^{-}(A) = - 0.3$

By routine computations, it is easy to see that λ^{μ} is a bipolar L-fuzzy sub ℓ - HX group of ϑ.

Definition 2.6. Let μ be a bipolar L - fuzzy subset defined on G. Let $\vartheta \subset 2^{G} - \{\phi\}$ be a ℓ - HX group on G. A bipolar L - fuzzy set λ^{μ} defined on ϑ is said to be a bipolar anti L - fuzzy sub ℓ - HX group on ϑ if for all A,B$\in\vartheta$.

i) $(\lambda^{\mu})^{+}(AB) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
ii) $(\lambda^{\mu})^{-}(AB) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
iii) $(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(A^{-1})$
iv) $(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(A^{-1})$
v) $(\lambda^{\mu})^{+}(A\vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
vi) $(\lambda^{\mu})^{-}(A\vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
vii) $(\lambda^{\mu})^{+}(A\wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
viii) $(\lambda^{\mu})^{-}(A\wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

where $(\lambda^{\mu})^{+}(A) = \min\{ \mu^{+}(x) / \text{for all } x\in A \subseteq G \}$
and
$$(\lambda^{\mu})^{-}(A) = \max\{ \mu^{-}(x) / \text{for all } x\in A\subseteq G \}$$

Example 2.2. Let $G=\{Z_{10}-\{0\}, \cdot_{10}\}$ be a group and define a bipolar L- fuzzy set μ on G as $\mu^{+}(1)=0.3$, $\mu^{+}(3)=0.7$, $\mu^{+}(7)=0.7$, $\mu^{+}(9)=0.8$ and $\mu^{-}(1)= - 0.4$, $\mu^{-}(3)= - 0.6$, $\mu^{-}(7)= - 0.6$, $\mu^{-}(9)= - 0.7$.

By routine computations, it is easy to see that μ is a bipolar anti L-fuzzy sub group of G.

Let $\vartheta = \{\{3, 7\}, \{1, 9\}\}$ be a ℓ - HX group of G.

Let us consider A = {3, 7}, B = {1, 9}.

$\cdot_{10}$	A	B
A	B	A
B	A	B

$\wedge$	A	B
A	A	B
B	B	B

$\vee$	A	B
A	A	A
B	A	B

Define$(\lambda^{\mu})^{+}$ (A) = min{μ^{+} (x) / for all x∈ A ⊆G }

and

$(\lambda^{\mu})^{-}$ (A) = max{μ^{-} (x) / for all x∈ A⊆ G }

Now $(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(\{3,7\}) = \min\{\mu^{+}(3), \mu^{+}(7)\} = \min\{0.7,0.7\} = 0.7$

$(\lambda^{\mu})^{+}(B) = (\lambda^{\mu})^{+}(\{1,9\}) = \min\{\mu^{+}(1), \mu^{+}(9)\} = \min\{0.3,0.8\} = 0.3$

$(\lambda^{\mu})^{+}(AB) = (\lambda^{\mu})^{+}(A) = 0.7$

$(\lambda^{\mu})^{+}(A\wedge B) = (\lambda^{\mu})^{+}(B) = 0.3$

$(\lambda^{\mu})^{+}(A\vee B) = (\lambda^{\mu})^{+}(A) = 0.7$

$(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(\{3,7\}) = \max\{\mu^{-}(3), \mu^{-}(7)\} = \max\{-0.6, -0.6\} = -0.6$

$(\lambda^{\mu})^{-}(B) = (\lambda^{\mu})^{-}(\{1,9\}) = \max\{\mu^{-}(1), \mu^{-}(9)\} = \max\{-0.4, -0.7\} = -0.4$

$(\lambda^{\mu})^{-}(AB) = (\lambda^{\mu})^{-}(A) = -0.6$

$(\lambda^{\mu})^{-}(A\wedge B) = (\lambda^{\mu})^{-}(B) = -0.4$

$(\lambda^{\mu})^{-}(A\vee B) = (\lambda^{\mu})^{-}(A) = -0.6$

By routine computations, it is easy to see that λ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ .

3. Properties of bipolar anti L-fuzzy sub ℓ - HX group

In this section, we discuss some of the properties ofbipolar anti L-fuzzy sub ℓ - HX group.

Theorem 3.1. Let G be a group. If μ is a bipolar anti L-fuzzy sub ℓ group of G then the bipolar L-fuzzy set λ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ .

Proof: Let μ be a bipolar anti L-fuzzy sub ℓ group of G and λ^{μ} be a bipolar L-fuzzy subset on G for any A,B∈ ϑ⊂G

i) $(\lambda^{\mu})^{+}$ (A) ∨ $(\lambda^{\mu})^{+}$ (B)=min{μ^{+} (x) / for all x∈ A ⊂G}∨ min{μ^{+} (y)/for all y∈ B⊂G}

= $\mu^{+}(x_0) \vee \mu^{+}(y_0)$, some $x_0 \in$ A, $y_0\in$ B and A,B⊂G

≥ $\mu^{+}(x_0\, y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

= min { μ^{+} (xy) / for all x∈ A, y∈ B and A,B⊂G}

= $(\lambda^{\mu})^{+}$ (AB).

So, $(\lambda^{\mu})^{+}$ (AB) ≤ $(\lambda^{\mu})^{+}$ (A) ∨ $(\lambda^{\mu})^{+}$ (B)

ii)$(\lambda^{\mu})^{-}$(A) ∧ $(\lambda^{\mu})^{-}$ (B) =max{ μ^{-} (x) / for all x∈ A ⊂G } ∧ max{ μ^{-} (y) / for all y∈ B⊂G }

= $\mu^{-}(x_0) \wedge \mu^{-}(y_0)$, some $x_0 \in$ A, $y_0\in$ B and A,B⊂G

$\leq \mu^-(x_0\, y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

$= \max\{ \mu^+(xy) \,/\, \text{for all } x\in A,\ y\in B \text{ and } A,B\subset G\}$

$= (\lambda^\mu)^-(AB)$

So, $(\lambda^\mu)^-(AB) \geq (\lambda^\mu)^+(A) \wedge (\lambda^\mu)^+(B)$

iii) $(\lambda^\mu)^+(A) = \min\{ \mu^+(x) \,/\, \text{for all } x\in A\subset G\}$

$= \min\{ \mu^+(x^{-1}) \,/\, \text{for all } x^{-1}\in A\subset G\}$

$= \min\{ \mu^+(x^{-1}) \,/\, \text{for all } x^{-1}\in A^{-1}\subset G\}$

So, $(\lambda^\mu)^+(A) = (\lambda^\mu)^+(A^{-1})$

iv) $(\lambda^\mu)^-(A) = \max\{ \mu^-(x) \,/\, \text{for all } x\in A\subset G\}$

$= \max\{ \mu^-(x^{-1}) \,/\, \text{for all } x^{-1}\in A\subset G\}$

$= \max\{ \mu^-(x^{-1}) \,/\, \text{for all } x^{-1}\in A^{-1}\subset G\}$

So, $(\lambda^\mu)^-(A) = (\lambda^\mu)^-(A^{-1})$

v) $(\lambda^\mu)^+(A) \vee (\lambda^\mu)^+(B) = \min\{ \mu^+(x) \,/\, \text{for all } x\in A\subset G \} \vee \min\{ \mu^+(y) \,/\, \text{for all } y\in B\subset G\}$

$= \mu^+(x_0) \vee \mu^+(y_0)$, some $x_0\in A$, $y_0\in B$ and $A,B\subset G$

$\geq \mu^+(x_0\vee y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

$= \min\{ \mu^+(x\vee y) \,/\, \text{for all } x\in A,\ y\in B \text{ and } A,B\subset G\}$

$= (\lambda^\mu)^+(A\vee B)$

So, $(\lambda^\mu)^+(A\vee B) \leq (\lambda^\mu)^+(A) \vee (\lambda^\mu)^+(B)$

vi) $(\lambda^\mu)^-(A)\wedge(\lambda^\mu)^-(B) = \max\{ \mu^-(x) \,/\, \text{for all } x\in A\subset G \} \wedge \max\{ \mu^-(y) \,/\, \text{for all } y\in B\subset G\}$

$= \mu^-(x_0) \wedge \mu^-(y_0)$, some $x_0\in A$, $y_0\in B$ and $A,B\subset G$

$\leq \mu^-(x_0\vee y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

$= \max\{ \mu^-(x\vee y) \,/\, \text{for all } x\in A,\ y\in B \text{ and } A,B\subset G\}$

$= (\lambda^\mu)^-(A\vee B)$

So, $(\lambda^\mu)^-(A\vee B) \geq (\lambda^\mu)^-(A) \wedge (\lambda^\mu)^-(B)$

vii) $(\lambda^\mu)^+(A)\vee(\lambda^\mu)^+(B) = \min\{ \mu^+(x) \,/\, \text{for all } x\in A\subset G \} \vee \min\{ \mu^+(y) \,/\, \text{for all } y\in B\subset G\}$

$= \mu^+(x_0) \vee \mu^+(y_0)$, some $x_0\in A$, $y_0\in B$ and $A,B\subset G$

$\geq \mu^+(x_0\wedge y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

$= \min\{ \mu^+(x\wedge y) \,/\, \text{for all } x\in A,\ y\in B \text{ and } A,B\subset G\}$

$= (\lambda^\mu)^+(A\wedge B)$

So, $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

viii) $(\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B) = \max\{\mu^{-}(x) \,/\, \text{for all } x \in A \subset G\} \wedge \max\{\mu^{-}(y) \,/\, \text{for all } y \in B \subset G\}$

$= \mu^{-}(x_0) \wedge \mu^{-}(y_0)$, some $x_0 \in A$, $y_0 \in B$ and $A, B \subset G$

$\leq \mu^{-}(x_0 \wedge y_0)$, μ is a bipolar anti L-fuzzy sub ℓ group on G

$= \max\{\mu^{-}(x \wedge y) / \text{for all } x \in A, y \in B \text{ and } A, B \subset G\}$

$= (\lambda^{\mu})^{-}(A \wedge B)$

So, $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence λ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ .

Theorem 3.2. Let $\lambda^{\mu} = ((\lambda^{\mu})^{+}, (\lambda^{\mu})^{-})$ be a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ then

i) $(\lambda^{\mu})^{+}(A) \geq (\lambda^{\mu})^{+}(E)$ and $(\lambda^{\mu})^{-}(A) \leq (\lambda^{\mu})^{-}(E)$ for all $A \in \vartheta$ and E is the identity element of ϑ.

ii) The subset $H = \{A \in \vartheta \,/\, (\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(E) \text{ and } (\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(E)\}$ is a sub ℓ - HX group of ϑ.

Proof i) Let $A \in \vartheta$

$(\lambda^{\mu})^{+}(E) = (\lambda^{\mu})^{+}(A A^{-1})$

$\leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(A^{-1})$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(A)$

$= (\lambda^{\mu})^{+}(A)$

Therefore, $(\lambda^{\mu})^{+}(A) \geq (\lambda^{\mu})^{+}(E)$, for all $A \in \vartheta$

Similarly, for all $A \in \vartheta$

$(\lambda^{\mu})^{-}(E) = (\lambda^{\mu})^{-}(A A^{-1})$

$\geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(A^{-1})$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(A)$

$= (\lambda^{\mu})^{-}(A)$

Therefore, $(\lambda^{\mu})^{-}(A) \leq (\lambda^{\mu})^{-}(E)$, for all $A \in \vartheta$

ii)Let $H = \{A \in \vartheta / (\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(E) \text{ and } (\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(E)\}$clearly, H is non-empty as $E \in H$ Let $A, B \in H$, then $(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(B) = (\lambda^{\mu})^{+}(E)$ and $(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(B) = (\lambda^{\mu})^{-}(E)$

Now, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B^{-1})$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$= (\lambda^{\mu})^{+}(E) \vee (\lambda^{\mu})^{+}(E)$

$= (\lambda^{\mu})^{+}(E)$

$(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(E)$

That is, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(E)$ and obviously $(\lambda^{\mu})^{+}(AB^{-1}) \geq (\lambda^{\mu})^{+}(E)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) = (\lambda^{\mu})^{+}(E)$ then $AB^{-1} \in H$

Now, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B^{-1})$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$= (\lambda^{\mu})^{-}(E) \wedge (\lambda^{\mu})^{-}(E)$

$= (\lambda^{\mu})^{-}(E)$

$(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(E)$

That is, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(E)$ and obviously $(\lambda^{\mu})^{-}(AB^{-1}) \leq (\lambda^{\mu})^{-}(E)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) = (\lambda^{\mu})^{-}(E)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) = (\lambda^{\mu})^{-}(E)$ then $AB^{-1} \in H$

Now, $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$= (\lambda^{\mu})^{+}(E) \vee (\lambda^{\mu})^{+}(E)$

$= (\lambda^{\mu})^{+}(E)$

$(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(E)$

That is, $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(E)$ and obviously $(\lambda^{\mu})^{+}(A \vee B) \geq (\lambda^{\mu})^{+}(E)$

Hence, $(\lambda^{\mu})^{+}(A \vee B) = (\lambda^{\mu})^{+}(E)$ then $A \vee B \in H$

Now, $(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$= (\lambda^{\mu})^{-}(E) \wedge (\lambda^{\mu})^{-}(E)$

$= (\lambda^{\mu})^{-}(E)$

$(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(E)$

That is, $(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(E)$ and obviously $(\lambda^{\mu})^{-}(A \vee B) \leq (\lambda^{\mu})^{-}(E)$

Hence, $(\lambda^{\mu})^{-}(A \vee B) = (\lambda^{\mu})^{-}(E)$ then $A \vee B \in H$

Now, $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$= (\lambda^{\mu})^{+}(E) \vee (\lambda^{\mu})^{+}(E)$

$= (\lambda^{\mu})^{+}(E)$

$(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(E)$

That is, $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(E)$ and obviously $(\lambda^{\mu})^{+}(A \wedge B) \geq (\lambda^{\mu})^{+}(E)$

Hence, $(\lambda^{\mu})^{+}(A \wedge B) = (\lambda^{\mu})^{+}(E)$ then $A \wedge B \in H$

Now, $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$= (\lambda^{\mu})^{-}(E) \wedge (\lambda^{\mu})^{-}(E)$

$= (\lambda^{\mu})^{-}(E)$

$(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(E)$

That is, $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(E)$ and obviously $(\lambda^{\mu})^{-}(A \wedge B) \leq (\lambda^{\mu})^{-}(E)$

Hence, $(\lambda^{\mu})^{-}(A \wedge B) = (\lambda^{\mu})^{-}(E)$ then $A \wedge B \in H$

Clearly, H is a sub ℓ - HX group of ϑ.

Theorem 3.3. Let $\lambda^{\mu}=((\lambda^{\mu})^{+},(\lambda^{\mu})^{-})$ be a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ with identity E. Then

i) $(\lambda^{\mu})^{+}(AB^{-1}) = (\lambda^{\mu})^{+}(E) \Rightarrow (\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(B)$, for all A,B in ϑ

ii) $(\lambda^{\mu})^{-}(AB^{-1}) = (\lambda^{\mu})^{-}(E) \Rightarrow (\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(B)$, for all A,B in ϑ

Proof: i) Let $\lambda^{\mu} =((\lambda^{\mu})^{+},(\lambda^{\mu})^{-})$be a bipolar anti L – fuzzy sub ℓ - HX group of a ℓ - HX group ϑ with identity E and $(\lambda^{\mu})^{+}(AB^{-1})= (\lambda^{\mu})^{+}(E)$ Then, for all A,B in ϑ.

$(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(A(B^{-1}B))$

$=(\lambda^{\mu})^{+}((A(B^{-1})B)$

$\leq (\lambda^{\mu})^{+}(AB^{-1}) \vee (\lambda^{\mu})^{+}(B)$

$=(\lambda^{\mu})^{+}(E) \vee (\lambda^{\mu})^{+}(B)$

$=(\lambda^{\mu})^{+}(B)$

That is, $(\lambda^{\mu})^{+}(A) \leq (\lambda^{\mu})^{+}(B)$

That is, $(\lambda^{\mu})^{+}(AB^{-1})=(\lambda^{\mu})^{+}(E) \Rightarrow (\lambda^{\mu})^{+}(A) \leq (\lambda^{\mu})^{+}(B)$

Since, $(\lambda^{\mu})^{+}(BA^{-1}) = (\lambda^{\mu})^{+}((BA^{-1})^{-1})$

$= (\lambda^{\mu})^{+}(AB^{-1})$

$= (\lambda^{\mu})^{+}(E)$

$(\lambda^{\mu})^{+}(BA^{-1}) = (\lambda^{\mu})^{+}(E)$

That is, $(\lambda^{\mu})^{+}(BA^{-1}) = (\lambda^{\mu})^{+}(E) \Rightarrow (\lambda^{\mu})^{+}(B) \leq (\lambda^{\mu})^{+}(A)$

Hence, $(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(B)$

Therefore, $(\lambda^{\mu})^{+}(AB^{-1}) = (\lambda^{\mu})^{+}(E) \Rightarrow (\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(B)$, for all A,B in ϑ.

ii)Let $\lambda^{\mu} =((\lambda^{\mu})^{+},(\lambda^{\mu})^{-})$ be a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ withidentity E and $(\lambda^{\mu})^{-}(AB^{-1})= (\lambda^{\mu})^{-}(E)$ Then, for all A,B in ϑ

$$
\begin{aligned}
(\lambda^{\mu})^{-}(A) &= (\lambda^{\mu})^{-}(A(B^{-1}B)) \\
&= (\lambda^{\mu})^{-}((A(B^{-1})B) \\
&\geq (\lambda^{\mu})^{-}(AB^{-1}) \wedge (\lambda^{\mu})^{-}(B) \\
&= (\lambda^{\mu})^{-}(E) \wedge (\lambda^{\mu})^{-}(B) \\
&= (\lambda^{\mu})^{-}(B)
\end{aligned}
$$

That is, $(\lambda^{\mu})^{-}(A) \geq (\lambda^{\mu})^{-}(B)$

That is, $(\lambda^{\mu})^{-}(AB^{-1}) = (\lambda^{\mu})^{-}(E) \Rightarrow (\lambda^{\mu})^{-}(A) \geq (\lambda^{\mu})^{-}(B)$

Since, $(\lambda^{\mu})^{-}(BA^{-1}) = (\lambda^{\mu})^{-}((BA^{-1})^{-1})$

$= (\lambda^{\mu})^{-}(AB^{-1})$

$= (\lambda^{\mu})^{-}(E)$

$(\lambda^{\mu})^{-}(BA^{-1}) = (\lambda^{\mu})^{-}(E)$

That is, $(\lambda^{\mu})^{-}(BA^{-1}) = (\lambda^{\mu})^{-}(E) \Rightarrow (\lambda^{\mu})^{-}(B) \geq (\lambda^{\mu})^{-}(A)$

Hence, $(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(B)$

Therefore, $(\lambda^{\mu})^{-}(AB^{-1}) = (\lambda^{\mu})^{-}(E) \Rightarrow (\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(B)$, for all A,B in ϑ.

4. Properties of lower level subsets of a bipolar anti l-fuzzy sub ℓ - HX group

In this section, We introduce the concept of lower level subsets of a bipolar ani L-fuzzy sub ℓ - HX group and discuss some of its properties.

Definition 4.1. Let λ^{μ} be a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ. For any $<\alpha,\beta> \in [0,1]$ x$[-1,0]$, We define the set $\lambda^{\mu}_{<\alpha,\beta>} = \{ A \in \vartheta / (\lambda^{\mu})^{+}(A) \leq \alpha$ and $(\lambda^{\mu})^{-}(B) \geq \beta \}$ is called the $< \alpha , \beta >$ lower level subset of λ^{μ}.

Theorem 4.1. Let λ^{μ} be a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ then for $< \alpha , \beta > \in [0,1]$ x$[-1,0]$ such that $(\lambda^{\mu})^{+}(E) \leq \alpha$, $(\lambda^{\mu})^{-}(E) \geq \beta$ and $\lambda^{\mu}_{<\alpha,\beta>}$is a sub ℓ - HX group of ϑ.

Proof: For all A,B$\in \lambda^{\mu}_{<\alpha,\beta>}$ we have $(\lambda^{\mu})^{+}(A) \leq \alpha$, $(\lambda^{\mu})^{-}(A) \geq \beta$ and $(\lambda^{\mu})^{+}(B) \leq \alpha$,

$(\lambda^{\mu})^{-}(B) \geq \beta$

Now $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$\leq \alpha \vee \alpha$

$= \alpha$

$\Rightarrow (\lambda^{\mu})^{+}(AB^{-1}) \leq \alpha$

$(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(\beta)$

$\geq \beta \wedge \beta$

$= \beta$

$\Rightarrow (\lambda^{\mu})^{-}(AB^{-1}) \geq \beta$

Hence, $AB^{-1} \in \lambda^{\mu}_{<\alpha,\beta>}$

Now $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$\leq \alpha \vee \alpha$

$= \alpha$

$\Rightarrow (\lambda^{\mu})^{+}(A \vee B \leq \alpha$

$(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$\geq \beta \wedge \beta$

$= \beta$

$\Rightarrow (\lambda^{\mu})^{-}(A \vee B) \geq \beta$

Hence, $A \vee B \in \lambda^{\mu}_{<\alpha,\beta>}$

Now $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

$\leq \alpha \vee \alpha$

$= \alpha$

$\Rightarrow (\lambda^{\mu})^{+}(A \wedge B) \leq \alpha$

$(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

$\geq \beta \wedge \beta$

$= \beta$

$\Rightarrow (\lambda^{\mu})^{-}(A \wedge B) \geq \beta$

Hence, $A \wedge B \in \lambda^{\mu}_{<\alpha,\beta>}$

Hence, $\lambda^{\mu}_{<\alpha,\beta>}$ is a sub ℓ - HX group of ϑ.

Definition 4.2. Let λ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of aℓ - HX group ϑ. The sub ℓ - HX groups $\lambda^{\mu}_{<\alpha,\beta>}$ for $< \alpha, \beta > \in [0,1] \times [-1,0]$ and $(\lambda^{\mu})^{+}(E) \leq \alpha$, $(\lambda^{\mu})^{-}(E) \geq \beta$ are called lower level sub ℓ - HX groups of λ^{μ}.

Theorem 4.2. Let ϑ be a ℓ - HX group and λ^{μ} be a bipolar anti L-fuzzy subset of ϑ such that $\lambda^{\mu}_{<\alpha,\beta>}$ is a sub ℓ - HX group of ϑ for $<\alpha,\beta> \in [0,1] \times [-1,0]$ such that $(\lambda^{\mu})^{+}(E) \leq \alpha$, $(\lambda^{\mu})^{-}(E) \geq \beta$ .Thenλ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ.

Proof: Let $A,B \in \vartheta$, let $A \in \lambda^{\mu}_{<\alpha 1, \beta 1>} \Rightarrow (\lambda^{\mu})^{+}(A) = \alpha_1$, $(\lambda^{\mu})^{-}(A) = \beta_1$ and $B \in \lambda^{\mu}_{<\alpha 2, \beta 2>} \Rightarrow (\lambda^{\mu})^{+}(B) = \alpha_2$, $(\lambda^{\mu})^{-}(B) = \beta_2$. Suppose $\lambda^{\mu}_{<\alpha 1, \beta 1>} < \lambda^{\mu}_{<\alpha 2, \beta 2>}$ then $A,B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$ As $\lambda^{\mu}_{<\alpha 2, \beta 2>}$ is a sub ℓ - HX group of ϑ, $AB^{-1} \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$,$A \wedge B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$ and $A \vee B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$,

Now $(\lambda^{\mu})^{+}(AB^{-1}) \leq \alpha_2$

$= \alpha_1 \vee \alpha_2$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) \geq \beta_2$
$= \beta_1 \wedge \beta_2$
$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $AB^{-1} \in \lambda^{\mu}$

Now $(\lambda^{\mu})^{+}(A \vee B) \leq \alpha_2$
$= \alpha_1 \vee \alpha_2$
$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
Hence, $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
Now $(\lambda^{\mu})^{-}(A \vee B) \geq \beta_2$
$= \beta_1 \wedge \beta_2$
$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $A \vee B \in \lambda^{\mu}$
Now $(\lambda^{\mu})^{+}(A \wedge B) \leq \alpha_2$
$= \alpha_1 \vee \alpha_2$
$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
Hence, $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$
Now $(\lambda^{\mu})^{-}(A \wedge B) \geq \beta_2$
$= \beta_1 \wedge \beta_2$
$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$
Hence, $A \wedge B \in \lambda^{\mu}$
Hence λ^{μ} is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ.

Theorem 4.3. Let ϑ be a ℓ - HX group and λ^{μ} be a bipolar anti L-fuzzy sub ℓ - HX group of ϑ. If two bipolar lower level sub ℓ - HX groups $\lambda^{\mu}_{<\alpha, \gamma>}$, $\lambda^{\mu}_{<\beta, \delta>}$ with $\alpha<\beta$ and $\delta<\gamma$ of λ^{μ} are equal if and only if there is no $A \in \vartheta$ such that $\alpha < (\lambda^{\mu})^{+}(A) \leq \beta$ and $\delta \leq (\lambda^{\mu})^{-}(A) < \gamma$.
Proof: Let $\lambda^{\mu}_{<\alpha, \gamma>} = \lambda^{\mu}_{<\beta, \delta>}$. Suppose that there exists $A \in \vartheta$ such that $\alpha < (\lambda^{\mu})^{+}(A) \leq \beta$ and $\delta \leq (\lambda^{\mu})^{-}(A) < \gamma$. Then $\lambda^{\mu}_{<\alpha, \gamma>} \subset \lambda^{\mu}_{<\beta, \delta>}$ since $A \in \lambda^{\mu}_{<\beta, \delta>}$ but not in $\lambda^{\mu}_{<\alpha, \gamma>}$ which contradicts the hypothesis. Hence there exists no $A \in \vartheta$ such that $\alpha < (\lambda^{\mu})^{+}(A) \leq \beta$ and $\delta \leq (\lambda^{\mu})^{-}(A) < \gamma$. Conversely, let there be no $A \in \vartheta$ such that $\alpha < (\lambda^{\mu})^{+}(A) \leq \beta$ and $\delta \leq (\lambda^{\mu})^{-}(A) < \gamma$. Since $\alpha < \beta$ and $\delta < \gamma$, we have $\lambda^{\mu}_{<\alpha, \gamma>} \subset \lambda^{\mu}_{<\beta, \delta>}$. Let $A \in \lambda^{\mu}_{<\beta, \delta>}$ then $(\lambda^{\mu})^{+}(A) \leq \beta$ and $(\lambda^{\mu})^{-}(A) \geq \delta$. since there exists no $A \in \vartheta$ such that $\alpha < (\lambda^{\mu})^{+}(A) \leq \beta$ and $\delta \leq (\lambda^{\mu})^{-}(A) < \gamma$, we have $(\lambda^{\mu})^{+}(A) \leq \alpha$ and $(\lambda^{\mu})^{-}(A) \geq \gamma$ which implies $A \in \lambda^{\mu}_{<\alpha, \gamma>}$ that is $\lambda^{\mu}_{<\beta, \delta>} \subset \lambda^{\mu}_{<\alpha, \gamma>}$. Hence, $\lambda^{\mu}_{<\alpha, \gamma>} = \lambda^{\mu}_{<\beta, \delta>}$.

Theorem 4.4. A L-fuzzy subset λ^{μ} of ϑ is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ if and only if the lower level subsets $\lambda^{\mu}_{<\alpha, \beta>}$, $<\alpha, \beta> \in$ Image λ^{μ} are sub ℓ - HX group of ϑ.
Proof. It is clear.

Theorem 4.5. Any sub ℓ - HX group H of a ℓ - HX group ϑ can be realized as a lower level sub ℓ - HX group of some bipolar anti L-fuzzy sub ℓ - HX group of ϑ.

Proof: Let $\lambda^{\mu} = ((\lambda^{\mu})^{+}, (\lambda^{\mu})^{-})$ be a bipolar L-fuzzy subset and $A\in\vartheta$,

Define $(\lambda^{\mu})^{+}(A)= \begin{cases} 0, & \text{if } A\in H \\ \alpha, & \text{if } A\notin H \end{cases}$ and

$(\lambda^{\mu})^{-}(A)= \begin{cases} \beta, & \text{if } A\in H \\ 0, & \text{if } A\notin H \end{cases}$

we shall prove λ^{μ} be a bipolar anti L-fuzzy sub ℓ - HX group of ϑ.

Let $A, B\in\vartheta$.

i) Suppose A, B∈H then AB∈H, AB^{-1}∈H, A∧B∈H and A∨B∈H

$(\lambda^{\mu})^{+}(A)= (\lambda^{\mu})^{+}(B)= 0$ and $(\lambda^{\mu})^{-}(A)= (\lambda^{\mu})^{-}(B)= \beta$

Now $(\lambda^{\mu})^{+}(AB^{-1}) = 0$

$\leq 0\vee 0$

$=(\lambda^{\mu})^{+}(A) \vee(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1})\leq (\lambda^{\mu})^{+}(A) \vee(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) = \beta$

$\geq\beta\wedge \beta$

$=(\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge(\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B) = 0$

$\leq 0\vee 0$

$=(\lambda^{\mu})^{+}(A) \vee(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B)\leq (\lambda^{\mu})^{+}(A) \vee(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B) = \beta$

$\geq\beta \wedge \beta$

$=(\lambda^{\mu})^{-}(A) \wedge(\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \geq (\lambda^{\mu})^{-}(A) \wedge(\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) = 0$

$\leq 0 \vee 0$

$=(\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B)\leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) = \beta$

$\geq\beta \wedge \beta$

$=(\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$.

ii) Suppose A∈H, B∉H then AB∉H, AB^{-1}∉H, A∧B∈H or A∧B∉H and A∨B∈H or A∨B∉H.

$(\lambda^{\mu})^{+}(A)= 0$, $(\lambda^{\mu})^{+}(B)= \alpha$ and $(\lambda^{\mu})^{-}(A)= \beta$, $(\lambda^{\mu})^{-}(B)= 0$

Define

- $(\lambda^{\mu})^{+}(A\wedge B) = \begin{cases} 0, & \text{if } A\wedge B \in H \\ \alpha, & \text{if } A\wedge B \notin H \end{cases}$ and

  $(\lambda^{\mu})^{-}(A\wedge B) = \begin{cases} \beta, & \text{if } A\wedge B \in H \\ 0, & \text{if } A\wedge B \notin H \end{cases}$

- $(\lambda^{\mu})^{+}(A\vee B) = \begin{cases} 0, & \text{if } A\vee B \in H \\ \alpha, & \text{if } A\vee B \notin H \end{cases}$ and

  $(\lambda^{\mu})^{-}(A\vee B) = \begin{cases} \beta, & \text{if } A\vee B \in H \\ 0, & \text{if } A\vee B \notin H \end{cases}$

Let $A\wedge B\in H$ and $A\vee B\in H$

Now $(\lambda^{\mu})^{+}(AB^{-1}) = \alpha$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) = 0$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B) = 0$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B) = \beta$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) = 0$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) = \beta$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$.

Let $A\wedge B\notin H$ and $A\vee B\notin H$

Now $(\lambda^{\mu})^{+}(AB^{-1}) = \alpha$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) = 0$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A \vee B) = \alpha$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$.

Hence, $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A \vee B) = 0$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A \wedge B) = \alpha$

$\leq 0 \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A \wedge B) = 0$

$\geq \beta \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$.

iii) Suppose $A, B \notin H$ then $AB^{-1} \in H$ or $AB^{-1} \notin H$, $A \wedge B \notin H$ and $A \vee B \notin H$

$(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(B) = \alpha$, $(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(B) = 0$, $(\lambda^{\mu})^{+}(A \wedge B) = (\lambda^{\mu})^{+}(A \vee B) = \alpha$ and $(\lambda^{\mu})^{-}(A \wedge B) = (\lambda^{\mu})^{-}(A \vee B) = 0$

Define

$$(\lambda^{\mu})^{+}(AB^{-1}) = \begin{cases} 0, & \text{if } AB^{-1} \in H \\ \alpha, & \text{if } AB^{-1} \notin H \end{cases} \quad \text{and}$$

$$(\lambda^{\mu})^{-}(AB^{-1}) = \begin{cases} \beta, & \text{if } AB^{-1} \in H \\ 0, & \text{if } AB^{-1} \notin H \end{cases}$$

Let $AB^{-1} \notin H$

Now $(\lambda^{\mu})^{+}(AB^{-1}) = \alpha$

$\leq \alpha \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) = 0$

$\geq 0 \wedge 0$

$= (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A \vee B) = \alpha$

$\leq \alpha \vee \alpha$

$= (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B) = 0$

$\geq 0 \wedge 0$

$=(\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) = \alpha$

$\leq \alpha\vee\alpha$

$=(\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) = 0$

$\geq 0\wedge 0$

$=(\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Thus in all cases, λ^{μ} be a bipolar anti L-fuzzy sub ℓ - HX group of ϑ. For this bipolar anti L-fuzzy sub ℓ - HX group, $\lambda^{\mu}_{<\alpha,\beta>}=H$

Remark. As a Consequence of the Theorem 4.3, Theorem 4.4 the lower level sub ℓ - HX groups of a bipolar anti L-fuzzy sub ℓ - HX group λ^{μ} of a ℓ - HX group ϑ form a chain. Since $(\lambda^{\mu})^{+}(E) \leq (\lambda^{\mu})^{+}(A)$ and $(\lambda^{\mu})^{-}(E) \geq (\lambda^{\mu})^{-}(A)$ for all A in ϑ. Therefore, $\lambda^{\mu}_{<\alpha 0,\beta 0>}$, $\alpha\in[0,1]$ and $\beta\in[-1,0]$. Where $(\lambda^{\mu})^{+}(E)=\alpha_0$, $(\lambda^{\mu})^{-}(E)=\beta_0$ is the smallest sub ℓ - HX group and we have the chain $\{E\} \subseteq \lambda^{\mu}_{<\alpha 0,\beta 0>} \subseteq \lambda^{\mu}_{<\alpha 1,\beta 1>} \subseteq \lambda^{\mu}_{<\alpha 2,\beta 2>} \subseteq \ldots\ldots \subseteq \lambda^{\mu}_{<\alpha n,\beta n>} = \vartheta$, where $\alpha_0 < \alpha_1 < \alpha_2 < \ldots\ldots < \alpha_n$ and $\beta_0 > \beta_1 > \beta_2 > \ldots. > \beta_n$.

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