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g^*s- Homeomorphism and Contra g^*s- Continuous Functions in Topological Space

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Abstract. In this paper, we introduced a new class of homeomorphism called gs homeomorphism and g*s homeomorphism. Also we investigate a new generalization of contra continuity called contra-g*s-continuous functions

Keywords: gs-homeomorphism, g*s-homeomorphism, Contra-g*s-continuous functions

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1. Introduction

Levine [7] introduction and investigated the concept of generalized closed sets in topological space .Arya and Nour[1] defined generalized semi open [briefly gs- open] sets using semi open sets. In 1987 Bhattacharya and Lahiri [3] introduced the class of semi – generalized closed sets (sg- closed sets) Balachandran [2] introduced generalized continuous maps in topological spaces. Homomorphism plays a very important role in topology.

In 1995, Maki et al. [4] introduced the concepts of semi – generalized homeomorphisms and generalized semi homeomorphisms and studied some semi topological properties. The notion of contra continuity was introduced and investigated by Dontchev [6] Dontchev and Nohiri [8] Jafari and Noiri [5] have introduced and investigated contra. Semi continuous, functions, contra – pre- continuous functions and contra -α-continuous functions between topological spaces.

Throughout this paper (X, τ) and (Y, σ) represents the non- empty topological spaces on which no reperation axiom are assumed unless otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

2. Preliminaries

Recall the following definitions.

Definition 2.1. A subset (X, τ) is said to be

(1) Semi-pre closed (β -closed)[6] set if int(cl(int(A))) $\subseteq$ A
(2) g-closed[6] set if cl(A) $\subseteq$ U , whenever A $\subseteq$ U and U is open in X
(3) w-closed[5] set if cl(A) $\subseteq$ U , whenever A $\subseteq$ U and U is semi-open in X
(4) α -closed[4] set if cl(int(cl(A))) $\subseteq$ A
(5) wg-closed[5] set if cl(int(A) $\subseteq$ U , whenever A $\subseteq$ U and U is open in X

(6) g*-closed[6] set if if cl(A)$\subseteq$U, whenever A$\subseteq$U and U is g-open in X

(7) g*s-closed[6] set if if scl(A)$\subseteq$U, whenever A$\subseteq$U and U is gs-open in X

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2. A map f:X$\to$Y is said to be

(1) Continuous function if f $^{-1}$(V) is closed in X for every closed set V in Y
(2) g-continuous function if f $^{-1}$(V) is g-closed in X for every closed set V in Y
(3) α-continuous function if f $^{-1}$(V) is α-closed in X for every closed set V in Y
(4) w-continuous function if f $^{-1}$(V) is w-closed in X for every closed set V in Y
(5) g*- continuous function if f $^{-1}$(V) is g*-closed in X for every closed set V in Y
(6) g*s- continuous function if f $^{-1}$(V) is g*s-closed in X for every closed set V in Y

Definition 2.3. A bijective function f: (X,τ)$\to$(Y,σ) is called

(1) homeomorphism if both f and f $^{-1}$ are continuous
(2) w-homeomorphism if both f and f $^{-1}$ are w-continuous
(3) g-homeomorphism if both f and f $^{-1}$ are g-continuous
(4) α-homeomorphism if both f and f $^{-1}$ are α-continuous
(5) g*- homeomorphism if both f and f $^{-1}$ are g*-continuous
(6) g*s- homeomorphism if both f and f $^{-1}$ are g*s-continuous

Definition 2.4. A map f:X$\to$Y is said to be

(1) Contra-continuous function if f $^{-1}$(V) is closed in X for every open set V in Y
(2) Contra-g-continuous function if f $^{-1}$(V) is g-closed in X for every open set V in Y
(3) Contra-α-continuous function if f $^{-1}$(V) is α-closed in X for every open set V in Y
(4) Contra-w-continuous function if f $^{-1}$(V) is w-closed in X for every open set V in Y
(5) Contra-g*- continuous function if f $^{-1}$(V) is g*-closed in X for every open set V in Y
(6) Contra-g*s- continuous function if f $^{-1}$(V) is g*s-closed in X for every open set V in Y

3. g*s-Homeomorphism

Definition 3.1. A bijection f:(X,τ)$\to$(Y,σ) is called g*s- homeomorphism if f and f $^{-1}$ are both g*s-continuous.

Example 3.2. Consider X=Y={a.b.c}, τ={X,ϕ,{a},{a,c}}, σ={Y, ϕ,{a},{b}}. Let the function f: X$\to$Y be an identity map. Then f is bijective Sb*-continuous and f $^{-1}$ is Sb*-continuous. Hence f is Sb*-homeomorphism.

Theorem 3.3. Every homeomorphism is g*s-homeomorphism but not conversely.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a homeomorphism. Since by the definition f and f $^{-1}$ is g*s -continuous . Then f is bijection. We know that every closed set is g*s -closed . Then every continuous function is g*s-continuous. Then f and f $^{-1}$ is g*s -continuous. Then f is g*s -homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 3.4. Consider X=Y={a,b,c} τ={X,ϕ,{a}}, σ={Y, ϕ,{a},{b}, {a,b}}. Let f:(X,τ)$\rightarrow$(Y, σ) be an identity map. Let A={a,c} is closed in Y and also it is g*s-closed in X . Then f is g*s-homeomorphism. But it is not a homeomorphism. Since {a,c} is not closed in X . f is not a homeomorphism.

Theorem 3.5. Every g*s -homeomorphism is sg -homeomorphism but not conversely
Proof: Let f:(X,τ)$\rightarrow$(Y,σ) be a g*s- homeomorphism . Since by the definition f anf f -1 is sg-continuous. Then f is bijection . We know that every g*s-closed set is sg-closed . Then every g*s-continuous function is sg-continuous. Then f and f -1 is sg-continuous. Then f is sg-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 3.6. Consider X=Y={a,b,c,} τ={X,ϕ,{b},{a,c}}, σ={Y, ϕ,{a,b}}. Let f:X $\rightarrow$Y be an identity map . Let A={a,c} is closed in Y and also it is sg-closed in X Then f is sg-homeomorphism. But it is not a g*s-homeomorphism. Since the inverse image {a,c} is not g*s-closed in X.

Theorem 3.7. Every g*s -homeomorphism is gs -homeomorphism but not conversely
Proof: Let f:(X,τ)$\rightarrow$(Y,σ) be a g*s- homeomorphism . Since by the definition f anf f^{-1} is gs-continuous. Then f is bijection. We know that every g*s-closed set is gs-closed. Then every g*s-continuous function is gs-continuous. Then f anf f^{-1} is gs-continuous. Then f is gs-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 3.8. Consider X=Y={a,b,c,d} τ={X,ϕ,{a},{b},{a,b}{a,b,c}}, σ={Y, ϕ,{a},{d},{c,d}, {a,c,d}}. Let f:X$\rightarrow$Y be an identity map .Let A={a,b} is closed in X and also it is gs-closed in Y. Then f is gs-homeomorphism. But it is not a g*s-homeomorphism. Since the inverse image {a,b} is not g*s-closed in X.

Theorem 3.9. Let f:(X,τ)$\rightarrow$(Y,σ) be a bijective g*s-continuous map,then the following statements are equivalent (i) f is an g*s -open map . (ii) f is an g*s -homeomorphism . (iii) f is an g*s -closed map.
Proof: (i) implies (ii) Let f^{-1}:(X,τ)$\rightarrow$(Y,σ) be a bijective g*s -continuous map .Let F be an closed map in (X,τ). Then X-F is open in (X,τ). Since f is g*s -open .f(X-F) is g*s -open in (Y, σ).f(F) is g*s -closed in (Y, σ). f is g*s -continuous. Now $((f^{-1})^{-1}(F))$ is g*s -closed in (Y, σ) . f^{-1} is g*s -continuous. Then f and f^{-1} is g*s -continuous. f is an g*s -homeomorphism (ii) implies (iii) Suppose f is an g*s -homeomorphism. By the definition f is bijective , f and f^{-1} are g*s -continuous .Let f be an g*s - closed set in (X, τ). Since f and f^{-1} are g*s -continuous. Then $(f^{-1})^{-1}$ (F)=f(F)is g*s -closed in (Y, σ) . Then f is g*s -closed map . (iiii) implies (i) Let f is an g*s -closed map. Let U is an g*s

-open in X . Then X-U is g*s -closed in Y . Since f is g*s -closed. f(X-U) is g*s -closed in Y. Y-f(U) is g*s -closed in Y . f(U) is g*s -open in Y . f is an g*s -open map .

Definition 3.10. A bijection f:(X, τ) $\rightarrow$ (Y, σ) is called g*s -irresolute if f $^{-1}$(V) is g*s -closed in (X, τ) for every g*s -closed V in (Y, σ).

Example 3.11. Consider X=Y={a.b.c.} , τ ={X, ϕ ,{a},{a,c}}, σ ={Y, ϕ ,{a}}. Let f: X $\rightarrow$ Y be an identity map . Let A={c} is g*s-clsoed in Y . Then f $^{-1}$({c})={c} is also g*s-closed in X . f is g*s-Irresolute.

Theorem 3.12. The composition of two g*s-Homeomorphisms need not be an g*s – Homeomorphism.

Proof: Let f : (X, τ) $\rightarrow$ (Y, σ) and g: (Y, σ) $\rightarrow$ (Z, ζ) be an g*s-Homeomorphism**.** By g*s-Homeomorphism, f and f $^{-1}$ are both g*s-continuous **.**We know that ,The composition of two continuous functions need not be a continuous function**.** Since the composition of two g*s-continuous functions need not be a g*s-continuous function.Therefore g $\circ$ f is need not be an g*s- homeomorphism.

Example 3.13. Let X=Y=Z={a,b,c,} τ ={X, ϕ ,{a},{a,c}}, σ ={Y, ϕ ,{a},{b},{a,b}}, ξ ={Z, ϕ ,{a}}.Let f and g be an identity map . Here f and g are g*s-Homeomorphism. But g $\circ$ f is not an g*s-homeomorphism, Since the inverse image of X in {b,c} is not g*s-closed in X.

Definition 3.14. A Space X is said to be g*s-compact if every cover of X by g*s-open sets has a finite sub cover.

Definition 3.15. Let x be a point of (X, τ) and V be a subset of X . Then V is called a g*s-neighborhood of x in (X, τ) if there exist a g*s-open set U of (X, τ) such that x $\in$ U $\subset$ V.

Definition 3.16. A topological space (X, τ) is called g*s-Hausdorff if for each pair x,y of distinct points of X, there exists g*s-neighborhoods U_1 and U_2 of x and Y respectively, that are disjoint.

Theorem 3.17. Let X be g*s-compact and set Y be a Hausdorff space. If f: (X, τ) $\rightarrow$ (Y, σ) is g*s-continuous , g*s-irresolute and bijective then f is g*s-homeomorphism.

Proof: Let A be a g*s-closed subset of the g*s-compact space X . Then A is g*s-compact. But f is g*s-irresolute . Hence f(A) is g*s-compact . Take g=f $^{-1}$. Then g $^{-1}$(A) is g*s-closed .We know that ,consequently g is an g*s-irresolute map. Then f $^{-1}$ is g*s-irresolute. f is g*s-homeomorphism.

Thoerem 3.18. If f : (X, τ) $\rightarrow$ (Y, σ) is a g*s-Homeomorphism then g*s-cl(f $^{-1}$(B))= f $^{-1}$(g*s-cl(B)) for all B $\subseteq$ Y is g*s-closed.

Proof: If $f : (X,\tau) \to (Y,\sigma)$ is a g*s-Homeomorphism. Since f is g*s-Homeomorphism. f and f^{-1} is both are g*s-irresolute. g*s-cl(f(B)) is closed in (Y,σ). f^{-1}(g*s-cl(f(B))) is g*s-closed in (X,τ). Thus g*s-cl(f^{-1}(B)) $\subseteq f^{-1}$(g*s-cl(B)).Again f^{-1} is irresolute . g*s-cl(f^{-1}(B)) is g*s-closed in (X,τ). $((f^{-1})^{-1})$ g*s-cl(f^{-1}(B))= f (g*s-cl(f^{-1}(B))) is g*s-closed in (X,τ). B= f (g*s-cl(f^{-1}(B))). Hence g*s-cl(f^{-1}(B))= f^{-1}(g*s-cl(B)).

Theorem 3.19. If f: $(X,\tau) \to (Y,\sigma)$ is an g*s-Homeomorphism then g*s-cl(f(B))=f(g*s-cl(B)) for all $B \subseteq X$.

Proof: If f: $(X,\tau) \to (Y,\sigma)$ is an g*s-Homeomorphism. Since f: $(X,\tau) \to (Y,\sigma)$ is an g*s-Homeomorphism. Then f^{-1}:$(Y,\sigma) \to (X,\tau)$ is also an g*s-Homeomorphism. Since f is an g*s-Homeomorphism then f and f^{-1} is both are g*s-Irresolute. (g*s-cl(f(B)) is g*s-closed in (Y,σ) . f^{-1} g*s-cl(f(B)) is g*s-closed in (X,τ). $(g^*s\text{-int}(A))^c = g^*s\text{-cl}(A^c)$. $(g^*s\text{-int}(B))^c = (g^*s\text{-cl}(B^c))^c$. Then $f(g^*s\text{-int}(B)=f((g^*s\text{-cl}(B^c))^c)=((g^*s\text{-cl}(B^c)))^c= g^*s\text{-cl}(f(B^c)))^c= g^*s\text{-int}(f(B))$. Therefore , g*s-cl(f(B))=f(g*s-cl(B)).

Theorem 3.20. The set g*s-h(X,τ) is a group under the composition of maps.

Proof: Define a binary operation * as follows . * : g*s-h$(X,\tau) \times$ g*s-h$(X,\tau) \to$ g*s-h(X,τ) f*g= $g \circ f$ for all f,g$\in$ g*s-h(X,τ). '$\circ$' is the usual operation of composition of maps $g \circ f \in$ g*s-h(X,τ). We Know That, the composition of maps is associative and the identity map.

I : $(X,\tau) \times (X,\tau) \in$ g*s-h(X,τ) serves as the identity element. If f$\in$ g*s-h(X,τ) then $f^{-1} \in$ g*s-h(X,τ) such that $f \circ f^{-1} = f^{-1} \circ f$ =I , and so inverse exists for each element of g*s-h(X,τ) is a group under composition of maps. g*s-h(X,τ) is a group under the composition of maps.

Thoerem 3.21. Let f: $(X,\tau) \to (Y,\sigma)$ be an g*s-Homeomorphism . Then f induces an isomorphism from the group g*s-h(X,τ) onto the group g*s-h(X,τ).

Proof: Let f: $(X,\tau) \to (Y,\sigma)$ be an g*s-Homeomorphism .We define I_f: g*s-h$(X,\tau) \to$ g*s-h(X,τ). Now 'f' induces an isomorphism from the group $I_f(h) = f \circ h \circ f^{-1}$ for every h $\in$ g*s-h(X,τ). Since I_f is a bijection .Further for every $h_1,h_2 \in$ g*s-h(X,τ). $I_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1}$. $I_f(h_1 \circ h_2) = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1})$. $I_f(h_1 \circ h_2) = I_f(h_1) * I_f(h_2)$. Thus I_f is a Homeomorphism and so it is an isomorphism induced by 'f'. f induces an isomorphism from the group g*s-h(X,τ) onto the group Sb*-h(X,τ).

4. Contra g^*s - continuous functions

In this section I introduce the concept of contra g^*s – continuous function in topological spaces.

Definition 4.1. A function $f : (X,\tau) \to (Y,\sigma)$ is called contra g^*s – continuous if the inverse image of every open set in Y is g^*s –closed in X.

Theorem 4.2. Every contra-continuous function is contra g^*s continuous but not conversely.

Proof: Let f : (X, τ) $\rightarrow$(Y, σ) be contra continuous. Let V be any open set in Y. Then the inverse image f^{-1} (V) is closed in X. since every closed set in g^*s – closed, f^{-1} (V) is g^*s –closed in X. Therefore f is contra g^*s – continuous.

Example 4.3. Consider X=Y={a,b,c,d} τ ={X, ϕ ,{a},{b},{a,b}{a,b,c}}, σ ={Y, ϕ ,{a},{a,b}}. Let f be an identity map. Here f is contra-g*s-continuous but not contra-continuous. Since the inverse image of {a} is not closed in X.

Theorem 4.4. Every contra g^*s continuous function is contra gs continuous function but not conversely.
Proof: Let f : (X, τ) $\rightarrow$(Y, σ) be contra g^*s continuous Let V be any open set in Y. then the inverse image f^{-1} (V) is g^*s closed in X. Since every g^*s –closed set is gs closed, f^{-1} (V) is gs- closed in X. Therefore f is contra-gs -continuous.

Example 4.5. Consider X=Y={a,b,c} τ ={X, ϕ ,{a},{a,c}}, σ ={Y, ϕ ,{a}} . Let f be an identity map. Here f is contra-gs-continuous but not contra-g*s-continuous. Since the inverse image of {a,b} is not g*s-closed in X.

Theorem 4.6. Every contra g^*s –continuous function is contra sg-continuous function but not conversely.
Proof: Let f : (X, τ) $\rightarrow$(Y, σ) be contra g^*s – continuous Let V be any open set in Y. then the inverse image f^{-1} (V) is g^*s –closed in X. since every g^*s-closed set is sg-closed, f^{-1} (V) is sg –closed in X. Therefore f is contra sg-continuous function.

Example 4.7. Consider X=Y={a,b} τ ={X, ϕ ,{b}}, σ ={Y, ϕ ,{a}} . Let f be an identity map. Here f is contra-sg-continuous but not contra-g*s-continuous. Since the inverse image of {a} is not g*s-closed in X.

Remark 4.8. Independentness of contra-g*s-continuity
(i) Contra-g*s continuous function is independent to contra-g-continuous function
(ii) Contra-g*s continuous function is independent to contra-g*-continuous function
(iii) Contra-g*s continuous function is independent to contra-w-continuous function
(iv) Contra-g*s continuous function is independent to contra-pre-continuous function.

The below examples proved the independentness of contra-g*s-continuity

Example 4.9. Consider X=Y={a,b,c,d} τ ={X, ϕ ,{a},{b},{a,b}{a,b,c}},. Let f:X$\rightarrow$Y be an identity map .Here f is contra-g*s-continuous but not contra-g-continuous. Since the inverse image of {b,c} is not g-closed in x. In this space σ ={Y, ϕ ,{a},{b},{a,b}, } and f be an identity map , f is contra-g-continuous but not contra-g*s-continuous . Since the inverse image of {a,b,d} is not g*s-closed in X.

Example 4.10. Consider X=Y={a,b,c} τ={X,ϕ,{a},{a,b}},. Let f:X$\rightarrow$Y be an identity map .Here f is contra-g*s-continuous but not contra-g*-continuous .since the inverse image of {b} is not g*-closed in x. In this space σ={Y, ϕ,{b},{a,c} } and f be an identity map, f is contra-g*-continuous but not contra-g*s-continuous . Since the inverse image of {a,c} is not g*s-closed in X.

Example 4.11. Consider X=Y={a,b,c,d} τ={X,ϕ,{a},{b},{a,b}{a,b,c}}. Let f: $\rightarrow$Y be an identity map. Here f is contra-g*s-continuous but not contra-pre continuous. Since the inverse image of {a} is not pre-closed in x. In this space σ ={Y, φ,{a},{b},{a,b}} and we define a map f(a)=b, f(b)=c, f(c)=a, f(d)=d . Here f is contra-pre-continuous but not contra-g*s-continuous . Since the inverse image of {c,d} is not g*s-closed in X.

Example 4.12. Consider X=Y={a,b,c,} τ={X,ϕ,{a},{a,c}}. Let f:X$\rightarrow$Y be an identity map .Here f is contra-g*s-continuous but not contra-w-continuous. Since the inverse image of {a,c} is not w-closed in x. In this space σ={Y, φ, {a},{a,b}} and f be an identity map. Here f is contra-w-continuous but not contra-g*s-continuous. Since the inverse image of {a,b} is not g*s-closed in X.

Remark 4.13. The composition of two contra-g*s-continuous functions need not be an contra-g*s-continuous function.

Example 4.14. Let X=Y=Z={a,b,c,d} τ={X, φ,{a},{b},{a,b},{a,b,c}}, σ={Y, φ,{a},{b},{a,b}}, ξ={Z, φ,{a}}.Let f and g be an identity map. Here f and g are g*s-Homeomorphism. But $g \circ f$ is not an g*s-homeomorphism. Since the inverse image of X in {a.c} is not g*s-closed in X.

Theorem 4.15. If a map f : (X,τ) $\rightarrow$(Y,σ) is g*s –irresolute map the g: (Y,σ) $\rightarrow$(Z, ζ) is g*s-continuous map then $g \circ f$: (X,τ) $\rightarrow$(Z,ζ) is contra-g*s-continuous function

Proof: Let F be an open set in (Z,ζ). Then g^{-1}(F) in g*s-closed in (Y,σ), because g is contra-g*s-continuous . Since f is g*s-irresolute, $f^{-1}(g^{-1}(F))=(g \circ f)^{-1}(F)$ id g*s-closed in X. Hence $g \circ f$ is contra-g*s-continuous function.

5. Conclusion

In this paper, we have introduced g*s-Homeomorphism, contra-g*s-continuous functions in topological spaces and studied some properties and this can be extended to other topological spaces like fuzzy and Bi-topological spaces. And these notions can be applied for investigating many other properties.

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