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On the Diophantine Equation $2^{2x+1} + 7^y = z^2$

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Abstract. In [7], it is shown that the Diophantine equation $4^x + 7^y = z^2$ has no solutions in non-negative integers. In this paper, investigating all odd powers of 2 with all even values of y, we establish that the title equation has only one solution when $x = 2$ and $y = 2$, whereas for all other values x no solutions exist.

Keywords: Diophantine equations, Catalan's Conjecture

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 2, 4, 7].

The general equation

$$p^x + q^y = z^2$$

has many forms. For the equation $4^x + 7^y = z^2$ it has been shown [7] that it has no solutions in positive integers. The equation

$$2^v + 7^y = z^2 \qquad (1)$$

when $v = 2x$ is even yields $4^x + 7^y = z^2$ as in [7]. In this paper, equation (1) with odd values $v = 2x + 1$ and even values $y = 2n$ is discussed. This is done in Section 2 utilizing the following conjecture.

In 1844 Catalan conjectured: The only solution in integers $r > 0$, $s > 0$, $a > 1$, $b > 1$ of the equation

$$r^a - s^b = 1$$

is $r = b = 3$ and $s = a = 2$.

The conjecture was proven by P. Mihăilescu [5] in 2002.

2. The main result

In this section, we consider the equation

$$2^{2x+1} + 7^{2n} = z^2$$

for all integers $x \geq 1$ and $n \geq 1$ where z is an odd positive integer. We determine all the solutions and no-solution cases for the equation. This is contained in the following Theorem 2.1.

Theorem 2.1. Let $x \geq 1$ and $n \geq 1$ be integers. Then the equation

$$2^{2x+1} + 7^{2n} = z^2 \tag{2}$$

has:

(a) No solutions when $x = 1$.
(b) Exactly one solution when $x = 2$.
(c) No solutions for all $x > 2$.

Proof: (a) Suppose that $x = 1$ in equation (2). We have

$$2^3 + 7^{2n} = z^2 \qquad \text{or} \qquad (z - 7^n)(z + 7^n) = 8.$$

Since $n \geq 1$, it clearly follows that the above right-hand equation is impossible.

Hence, when $x = 1$ equation (2) has no solutions.

(b) Suppose that $x = 2$. When $n = 1$, then

$$2^5 + 7^2 = 9^2 \tag{3}$$

is a solution of equation (2).

We now show that solution (3) is the only solution of equation (2) when $x = 2$.

Suppose that there exists a value $n = t > 1$ with odd $z > 9$ satisfying

$$2^5 + 7^{2t} = z^2 \qquad \text{or} \qquad 2^5 = (z - 7^t)(z + 7^t). \tag{4}$$

Since $z - 7^t > 1$ whereas $z + 7^t > 49$, it follows that equation (4) is impossible.

Our supposition that when $x = 2$ there exist values $n = t > 1$ and $z > 9$ which satisfy the equation is therefore false.

Hence, solution (3) is the only solution when $x = 2$.

The solution is unique.

(c) Suppose that $x > 2$. Equation (2) implies

$$2^{2x+1} = z^2 - 7^{2n} = (z - 7^n)(z + 7^n).$$

Let α, β be positive integers. Denote

$$z - 7^n = 2^\alpha, \qquad z + 7^n = 2^\beta, \qquad \alpha < \beta, \qquad \alpha + \beta = 2x + 1. \tag{5}$$

From (5) it follows that $2 \cdot 7^n = 2^\beta - 2^\alpha = 2^\alpha(2^{\beta-\alpha} - 1)$. Since 7^n and $2^{\beta-\alpha} - 1$ are both odd, therefore $\alpha = 1$. Hence

$$2^{\beta-1} - 7^n = 1. \tag{6}$$

In (5), the value $\alpha = 1$ yields $\beta = 2x$ and since $x > 2$ it follows that $\beta > 4$. The value $n = 1$ in (6) implies that $2^{\beta-1} = 8$, and hence $\beta = 4$ which is impossible. Therefore $n > 1$ in (6). By Catalan's Conjecture it now follows that (6) has no solutions.

Thus, when $x > 2$ equation (2) has no solutions.

This concludes the proof of Theorem 2.1. □

3. Conclusion

In this paper we have established that the equation $2^{2x+1} + 7^{2n} = z^2$ has a unique solution when $x = 2$, $n = 1$, $z = 9$. Suppose that $y = 1$ is fixed in $2^v + 7^y = z^2$. It has been shown [3] that for all values $v \geq 1$, $2^v + 7 = z^2$ has a unique solution when $v = 1$, namely $2 + 7 = 3^2$. The following two questions concerning $2^v + 7^y = z^2$ may now be raised.

Question 1. If $v = 1$, does there exist an odd value $y > 1$ satisfying $2 + 7^y = z^2$? The values $y = 3, 5, 7, 9$ yield a negative answer.

Question 2. If $v > 1$ is odd, does there exist an odd value $y > 1$ satisfying $2^v + 7^y = z^2$?
Each of the values $v = 3, 5, 7$ with the three respective values $y = 3, 5, 7$ yields a negative answer.

We presume that the answers to Questions 1 and 2 are negative.

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