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Some Results in Generalization of Janowski Functions Associated with (*j*,*k*)-Symmetric Points

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Abstract. The objective of the present paper is to study results that are defined using notions of generalized Janowski functions and (j,k) -symmetrical functions. In particular, we derive integral representations and study the covering theorem also the quotient of analytical representations of starlikeness and convexity with respect to j,k -symmetric points, we will study the expression $\dfrac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} = \dfrac{1+zf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)}$.

Keywords: Janowski functions, Subordination, Starlike functions, Convex functions, (j,k) -symmetric points.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive

Definition 1.1. *[2] Let* $\mathcal{P}$ *denote the class of analytic functions of the form* $p(z)=1+\sum_{n=1}^{\infty}p_n z^n$ *defined on* $\mathcal{U}$ *and satisfying* $p(0)=1$, $Rep(z)>0$, $z\in\mathcal{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z)=\frac{1+w(z)}{1-w(z)}$, where $w\in\Omega$ and

$$\Omega=\{w\in\mathcal{A}: w(0)=0, |w(z)|<1\}. \tag{1.2}$$

The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

Like in [1], let $\mathcal{P}[A,B]$, with $-1\leq B<A\leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z)=\frac{1+Aw(z)}{1+Bw(z)}$, $z\in\mathcal{U}$, where $w\in\Omega$. Remark that $p\in\mathcal{P}[A,B]$ if and only if $p(z)\prec\frac{1+Az}{1+Bz}$.

In [13] the class $\mathcal{P}[A,B,\alpha]$ of generalized Janowski functions was introduced. For arbitrary numbers $A,B,\alpha,$ with $-1\leq B<A\leq 1$, $0\leq\alpha<1$, a function p analytic in $\mathcal{U}$ with $p(0)=1$ is in the class $\mathcal{P}[A,B,\alpha]$ if and only if $p(z)\prec\frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}$.

In our work we define the class $\mathcal{P}[A,B,\alpha,\beta]$ of generalized Janowski functions was introduced. For arbitrary numbers A,B,α,β with $-1\leq B<A\leq 1$, $0\leq\alpha,\beta<1$, and $\alpha+\beta<1$ a function p analytic in $\mathcal{U}$ with $p(0)=1$ is in the class $\mathcal{P}[A,B,\alpha,\beta]$ if and only if

$$p(z)\prec\frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z}\Leftrightarrow p(z)=\frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+[(1-\beta)B+\beta A]w(z)},\ w\in\Omega.$$

A function f is belongs to the class $\mathcal{S}^*[A,B,\alpha,\beta]$ if $\frac{zf'(z)}{f(z)}\in\mathcal{P}[A,B,\alpha,\beta]$.

In order to define a new class of generalized Janowski symmetrical functions defined in the open unit disk $\mathcal{U}$, we first recall the notion of k-fold symmetric functions defined in k-fold symmetric domain, where k is any positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$

carries $\mathbb{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathbb{D}$ if for every z in $\mathbb{D}$ we have

$$f\left(e^{\frac{2\pi i}{k}}z\right)=e^{\frac{2\pi i}{k}}f(z),\ z\in\mathbb{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k=2$ we get class of odd univalent functions. In 1995, Liczberski and Polubinski [19] constructed the theory of (j,k)-symmetrical functions for $j=0,1,2,\ldots,k-1)$ and $(k=2,3,\ldots$. If $\mathbb{D}$ is k-fold symmetric domain and j any integer, then a function $f:\mathbb{D}\to\mathbb{C}$ is called (j,k)-symmetrical if for each $z\in\mathbb{D}$, $f(\varepsilon z)=\varepsilon^j f(z)$. We note that the (j,k)-symmetrical functions is a generalization of the notions of even, odd, and k-symmetrical functions

The theory of (j,k)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [19].

Denote the family of all (j,k)-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem:

Theorem 1. [19, Page 16] For every mapping $f:\mathcal{U}\mapsto\mathbb{C}$, and a k-fold symmetric set $\mathcal{U}$, there exists exactly one sequence of (j,k)-symmetrical functions $f_{j,k}$ such that

$$f(z)=\sum_{j=0}^{k-1}f_{j,k}(z),$$

where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f\left(\varepsilon^{v}z\right),\ z\in\mathcal{U}. \tag{1.3}$$

Al Sarari and Latha [3] introduced and studied the classes $\mathcal{S}^{(j,k)}(A,B)$ and $\mathcal{K}^{(j,k)}(A,B)$ which are starlike and convex with respect to (j,k)-symmetric points. For more details about the classes with (j,k)-symmetrical functions see [9, 10, 14].

Definition 1.2. *A function* $f\in\mathcal{A}$ *is said to belongs to the class* $\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, *with* $-1\le B<A\le 1$, $0\le\alpha,\beta<1$, *and* $\alpha+\beta<1$ *if*

$$\frac{zf'(z)}{f_{j,k}(z)}\prec\frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z},$$

where $f_{j,k}$ are defined by (1.3).

We note that special values of j,k,A,B,α and β yield the following classes:
For $\alpha=\beta=0$ we get the class introduced and studied by Alsarari and Latha [3]
For $j=1$ and $\alpha=\beta=0$ the class studied by Ohsang K and Yaungjae [5].
For $j=k=A=-B$ and $\beta=0$ the class introduced by Polatoglu, Bolcal, Sen and Yavuz, [13].

For $j=A=-B=1,\alpha=0$ and $\beta=0$ the class is studied by Sakaguchi [16], etc. The second and third authors studied some classes with (j,k)-symmetrical functions [3, 9, 10, 14]. In this paper we will study the covering theorem also the quotient of analytical representations of starlikeness and convexity with respect to j,k-symmetric points, we will study the expression

$$\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)}=\frac{1+zf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)}$$

We need the following lemmas to prove our main results:

Lemma 1.3. *[4] Let Ω be a subset of the complex plan $\mathbb{C}$ and let the function $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ satisfy $\psi(Me^{i\theta},Ne^{i\theta};z)\notin\Omega$ for all real θ, $N\ge M$ and for all $z\in U$, if the function $p(z)$ is analytic in U, $p(0)=0$ and $\psi(p(z),zp'(z);z)\in\Omega$ for all $z\in U$ then $|p(z)|<M$, $z\in U$.*

2. Main results

Theorem 2.1. *Let $f\in S^{(j,k)}[A,B,\alpha,\beta]$, with $-1\le B<A\le 1$, $0\le\alpha,\beta<1$, and $\alpha+\beta<1$. Then*

$$f_{j,k}(z)=z.exp\left((A-B)(1-\alpha-\beta)\frac{1}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{\widetilde{w(t)}}{t(1+[(1-\beta)B+\beta A]\widetilde{w(t)})}dt\right),\quad z\in U, \tag{2.1}$$

where $\widetilde{w(z)}\in\Omega$.

Proof: Suppose that $f\in S^{(j,k)}[A,B,\alpha,\beta]$.

$$\frac{zf'(z)}{f_{j,k}(z)}\prec\frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z}, \tag{2.2}$$

and

$$\frac{zf'(z)}{f_{j,k}(z)}=\frac{1+[(1-\alpha)A+\alpha B]\widetilde{w(z)}}{1+[(1-\beta)B+\beta A]\widetilde{w(z)}}, \tag{2.3}$$

for some $\widetilde{w(z)}\in\Omega$. Substituting z by $\varepsilon^v z$ in (2.3), it follows

$$\frac{\varepsilon^{v} z f'(\varepsilon^{v} z)}{f_{j,k}(\varepsilon^{v} z)} = \frac{1+[(1-\alpha)A+\alpha B]\,\widetilde{w(\varepsilon^{v} z)}}{1+[(1-\beta)B+\beta A]\,\widetilde{w(\varepsilon^{v} z)}}, \quad (z \in \mathrm{U}), \tag{2.4}$$

Letting $v = 0,1,2,\ldots,k-1$ in (2.4) respectively, and summing them we have

$$\frac{z\dfrac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{v-vj} f'(\varepsilon^{v} z)}{f_{j,k}(z)} = \frac{1}{k}\sum_{v=0}^{k-1}\frac{1+[(1-\alpha)A+\alpha B]\,\widetilde{w(\varepsilon^{v} z)}}{1+[(1-\beta)B+\beta A]\,\widetilde{w(\varepsilon^{v} z)}}, \tag{2.5}$$

that is

$$\frac{f_{j,k}'(z)}{f_{j,k}(z)} - \frac{1}{z} = (A-B)(1-\alpha-\beta)\frac{1}{k}\sum_{v=0}^{k-1}\frac{\widetilde{w(\varepsilon^{v} z)}}{z(1+[(1-\beta)B+\beta A]\,\widetilde{w(\varepsilon^{v} z)})}. \tag{2.6}$$

Integrating get (2.1)

Theorem 2.2. *Let* $f \in \mathrm{S}^{(j,k)}[A,B,\alpha,\beta]$, *with* $-1 \le B < A \le 1$, $0 \le \alpha,\beta < 1$, *and* $\alpha+\beta < 1$. *Then*

$$f(z) = \int_0^z exp\left((A-B)(1-\alpha-\beta)\frac{1}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^{v}\xi}\frac{\widetilde{w(t)}}{t(1+[(1-\beta)B+\beta A]\,\widetilde{w(t)})}dt\right)\cdot\left[\frac{1+[(1-\alpha)A+\alpha B]w(\xi)}{1+[(1-\beta)B+\beta A]w(\xi)}\right]d\xi, \tag{2.7}$$

for $z \in \mathrm{U}$ and some $w(z) \in \Omega$.

Proof: Suppose that $f \in \mathrm{S}^{(j,k)}[A,B,\alpha,\beta]$.

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z}, \tag{2.8}$$

we have

$$zf'(z) = f_{j,k}(z)\frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+[(1-\beta)B+\beta A]w(z)},$$

and by Theorem 2.1 we have

$$f'(z) = exp\left((A-B)(1-\alpha-\beta)\frac{1}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^{v} z}\frac{\widetilde{w(t)}}{t(1+[(1-\beta)B+\beta A]\,\widetilde{w(t)})}dt\right)\frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+[(1-\beta)B+\beta A]w(z)}. \tag{2.9}$$

Integrating get (2.7).

Theorem 2.3. *Let* $f \in \mathrm{A}$, $k \ge 2$, $j = 0,1,2,\ldots,k-1$ *are a natural numbers,*

$1 \le B < A \le 1, 0 \le \alpha, \beta < 1$ *and* $\alpha + \beta < 1$. *Also, let* $\Omega = \mathbb{C} \setminus \Omega_1$, *where*

$$\Omega_1 = \left\{ 1 + N(1-\alpha-\beta)(A-B) \times \frac{f_{j,k}(z)}{zf'_{j,k}(z)} \times \frac{e^{i\theta}}{(1+[(1-\alpha)A+\alpha B]e^{i\theta})(1+[(1-\beta)B+\beta A]e^{i\theta})} : z \in \mathbb{U}, \theta \in \mathbb{R}, N \ge 1 \right\}$$

if

$$\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} \in \Omega, \quad z \in \mathbb{U},$$

then $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$.

Proof: Let us define the functions

$$p(z) = \frac{zf'(z)}{f_{j,k}(z)}, \qquad and \qquad p(z) = \frac{1+[(1-\alpha)A+\alpha B]q(z)}{1+[(1-\beta)B+\beta A]q(z)},$$

we note that both functions are analytic in U and $p(0) - 1 = q(0) = 0$, using Lemma 1.3 with $N = 1$ and

$$\psi(r, s; z) = 1 + (A-B)(1-\alpha-\beta) \times \frac{f_{j,k}(z)}{zf'_{j,k}(z)} \times \frac{s}{(1+[(1-\alpha)A+\alpha B]r)(1+[(1-\beta)B+\beta A]r)}.$$

Since $|q(z)| < 1, z \in \mathbb{U}$, we have

$$\psi(q(z), zq'(z); z) = \frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)}. \tag{2.10}$$

Equation (2.10) is equivalent to the following subordinations,

$$w(z) = \frac{1-p(z)}{[(1-\beta)B+\beta A]p(z) - [(1-\alpha)A+\alpha B]} \prec z \quad and \quad p(z) \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z},$$

which proves that $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$.

Theorem 2.4. *Let* $f \in \mathcal{A}, k \ge 2, j = 0,1,2,..., k-1$ *are a natural numbers,* $1 \le B < A \le 1, 0 \le \alpha, \beta < 1$ *and* $\alpha + \beta < 1$. *Also let*

$$\lambda \equiv \begin{cases} \dfrac{(A-B)(1-\alpha-\beta)\mu}{(1+[(1-\alpha)A+\alpha B])(1+[(1-\beta)B+\beta A])}, & 0 \le [(1-\beta)B+\beta A] < [(1-\alpha)A+\alpha B], \\ \dfrac{(A-B)(1-\alpha-\beta)\mu}{(1-[(1-\alpha)A+\alpha B])(1-[(1-\beta)B+\beta A])}, & [(1-\beta)B+\beta A] < [(1-\alpha)A+\alpha B] \le 0, \\ \dfrac{2\mu\sqrt{|[(1-\alpha)A+\alpha B]||[(1-\beta)B+\beta A]|}}{1-[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A]}, & [(1-\beta)B+\beta A] < 0 < [(1-\alpha)A+\alpha B], and \\ & |((1-\alpha)A+\alpha B+[(1-\beta)B+\beta A])(1+[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A])| \\ & \le 4|[(1-\alpha)A+\alpha B]||[(1-\beta)B+\beta A]|. \end{cases}$$

If

$$\left|\frac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \frac{1}{\mu}$$

and

$$\left|\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} - 1\right| < \lambda$$

for all $z \in \mathrm{U}$ then $f \in \mathrm{S}^{(j,k)}[A, B, \alpha, \beta]$.

Proof: Let us define $\Omega_2 = \{w : |w-1| < \lambda\}$ a set defined in the complex plan C.

In view of Theorem 2.3, to prove this theorem it is enough to show that $\Omega_2 \cap \Omega_1 = \phi$. If $w \in \Omega_1$ then for some $z \in \mathrm{U}$, $\theta \in \mathrm{R}$ and $N \geq 1$, we have

$$|w-1| = \left| N(A-B)(1-\alpha-\beta) \times \frac{f_{j,k}(z)}{zf'_{j,k}(z)} \times \frac{e^{i\theta}}{(1+[(1-\alpha)A+\alpha B]e^{i\theta})(1+[(1-\beta)B+\beta A]e^{i\theta})} \right|$$

$$> \frac{(A-B)(1-\alpha-\beta)\times\mu}{|1+[(1-\alpha)A+\alpha B]e^{i\theta}|.|1+[(1-\beta)B+\beta A]e^{i\theta}|}$$

$$= \frac{(A-B)(1-\alpha-\beta).\mu}{\sqrt{1+[(1-\alpha)A+\alpha B]^2+2[(1-\alpha)A+\alpha B]t}\times[\sqrt{1+[(1-\beta)B+\beta A]^2+2[(1-\beta)B+\beta A]t}]} \equiv h(t),$$

where $t = \cos\theta \in [-1,1]$. If we show that $h(t) \geq \lambda$ for all $t \in [-1,1]$.

We will split this proof in the next three cases:

Case 1. For $0 \leq [(1-\beta)B+\beta A] < [(1-\alpha)A+\alpha B]$, and

$$h(t) \geq h(1) = \frac{(A-B)(1-\alpha-\beta)\mu}{(1+[(1-\alpha)A+\alpha B])(1+[(1-\beta)B+\beta A])} = \lambda$$

and $t \in [-1,1]$.

Case 2. For, $[(1-\beta)B+\beta A] < [(1-\alpha)A+\alpha B] \leq 0$ then

$$h(t) \geq h(-1) = \frac{(A-B)(1-\alpha-\beta)\mu}{(1-[(1-\alpha)A+\alpha B])(1-[(1-\beta)B+\beta A])} = \lambda$$

and $t \in [-1,1]$.

Case 3. For $[(1-\beta)B+\beta A] < 0 < [(1-\alpha)A+\alpha B]$. Then the function $h(t)$ attains its minimal value for

$$t_* = -\frac{[(1+\beta-\alpha)A+(1+\alpha-\beta)B](1+[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A])}{4[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A]} \in [-1,1]$$

if and only if

$$\begin{aligned} &4[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A] \\ &\quad \leq \{(1+\beta-\alpha)A+(1+\alpha-\beta)B\}\{1+[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A]\} \\ &\quad\quad \leq 4[(1-\alpha)A+\alpha B]\,|[(1-\beta)B+\beta A]|. \end{aligned}$$

That value is $h(t_*) = \dfrac{2\mu\sqrt{[(1-\alpha)A+\alpha B]\,|\,[(1-\beta)B+\beta A]\,|}}{1-[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A]} = \lambda.$ It will imply that $w \notin \Omega_2$ and the proof is complete.

For $A = 1-2\gamma$ $(0 \le \gamma < 1), \alpha = \beta = 0$ and $B = -1$ in Theorem 2.4 we get the following corollary.

Corollary 2.5. *Let* $f \in \mathrm{A}, k \ge 2$ *is a natural number,* $j = 0,1,2,..,k-1$ $(0 \le \gamma < 1)$ and $\mu > 1$. Also let

$$\lambda_1 \equiv \begin{cases} \mu/2, & \text{if } 0 \le \gamma \le 1/2, \\ \dfrac{1-\gamma}{2\gamma}\mu, & \text{if } 1/2 < \gamma < 1. \end{cases}$$

If

$$\left|\frac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \frac{1}{\mu} \quad and \quad \left|\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} - 1\right| < \lambda_1$$

and for all $z \in \mathrm{U}$ then $f \in \mathrm{S}^{(j,k)}[\gamma]$.

Proof: First, let $\lambda_1 = \dfrac{(1-\gamma)\mu}{1+|1-2\gamma|} = \dfrac{(A-B)(1-\alpha-\beta)\mu}{(1+|[(1-\alpha)A+\alpha B]|)(1+|[(1-\beta)B+\beta A]|)}$.

Further, if $0 \le \gamma \le \dfrac{1}{2}$ then

$[(1-\alpha)A+\alpha B] \ge 0, [(1-\alpha)A+\alpha B][(1-\beta)B+\beta A] \ge 0$ and the conclusion of the Corollary follows since $\lambda_1 = \lambda$. In the case when $\dfrac{1}{2} < \gamma < 1$, we have $[(1-\alpha)A+\alpha B] > 0$ and $[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A] < 0,$ but

$$\{(1+\beta-\alpha)A+(1+\alpha-\beta)B\}\{1+[(1-\alpha)A+\alpha B][(1-\beta)B+\beta A]\}$$
$$= -4\gamma^2 > 4(1-2\gamma) = 4[(1-\alpha)A+\alpha B]\,|\,[(1-\beta)B+\beta A]\,|.$$

Again the conclusion follows because of $\lambda_1 = \lambda$.

Putting $\gamma = 0$ in Corollary 2.5 we obtain the following result.

Corollary 2.6. *Let* $f \in \mathrm{A}, k \ge 2$ *is a natural number,* $j = 0,1,2,..,k-1,$ *and* $\mu > 1$. *Also, let* $|zf'_{j,k}(z)/f_{j,k}(z)| > 1/\mu$ *for all* $z \in \mathrm{U}$.

If

$$\left|\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} - 1\right| < \frac{\mu}{2}$$

for all $z \in \mathrm{U}$ then $f \in \mathrm{S}^{(j,k)}[0] = \mathrm{S}^{(j,k)}[1,1] = \mathrm{S}^{(j,k)}$.

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