Annals of Pure and Applied Mathematics
Vol. 16, No. 1, 2018, 203-214
ISSN: 2279-087X (P), 2279-0888(online)
Published on 15 January 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v16n1a22

Annals of
Pure and Applied Mathematics

q-Analogue of Univalent Functions with Negative Coefficients

***Hamid Shamsan*[1] and *S. Latha*[2]**

Department of Mathematics, Yuvaraja's College, University of Mysore
Mysore 570 005, INDIA
[1]hmas19771@gmail.com, [2]drlatha@gmail.com
[1]Corresponding author

Received 6 December 2017; accepted 26 December 2017

Abstract. In this paper, we derive a necessary and sufficient condition, distortion theorem and coefficient inequalities are determined for univalent functions with negative coefficients that are q-starlike of order β and q-convex of order β. We also establish extreme point results, some results concerning the partial sums for the function $f(z)$ belonging to the class $T_q^*(\beta)$.

Keywords: Univalent functions, Starlike, Convex, q-Derivative, Partial sums, Extreme points.

AMS Mathematics Subject Classification (2010): 30C45

1. Introduction

Let S denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Which are analytic and univalent in the open unit disk

$$\mathsf{U} = \{z : z \in \mathsf{C} \ and \ |z| < 1\},$$

with $S^*(\beta)$ and $K(\beta)$, $0 \le \beta < 1$, designating the subclasses of S consisting of functions starlike and convex of order β. We shall denote by T the subclass of S consisting of functions that may be expressed in the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \ge 0). \tag{1.2}$$

Further, let

$$T_q^*(\beta) = S_q^*(\beta) \cap T \qquad (0 \le \beta < 1) \tag{1.3}$$

and

$$C_q(\beta) = K_q(\beta) \cap T \qquad (0 \le \beta < 1). \tag{1.4}$$

Jackson[2] initiated q-calculus and developed the concept of the q-integral and q-derivative.

For a function $f \in \mathcal{S}$ given by (1.1) and $0 < q < 1$, the q-derivative of f is defined by

Definition 1.1.

$$\partial_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0, \qquad 0 < q < 1. \end{cases} \tag{1.5}$$

Equivalently (1.5), may be written as $\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$ where

$[n]_q = \dfrac{1-q^n}{1-q}$. Note that as $q \to 1$, $[n]_q \to n$.

Making use of $\partial_q f$ T. M. Seoudy and M. K. Aouf [4] introduced the subclasses $S_q^*(\beta)$ and $K_q(\beta)$ defined by

Definition 1.2. *A function $f(z) \in \mathcal{A}$ is said to be q-starlike of order β, $0 \leq \beta < 1$, if and only if* $Re\left\{\dfrac{z\partial_q f(z)}{f(z)}\right\} > \beta, \qquad$ *for all* $z \in \mathcal{U}$.

We denote by $S_q^*(\beta)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order β in the unit disk $\mathcal{U}$.

Definition 1.3. *A function $f(z) \in \mathcal{A}$ is said to be q-convex of order β, $0 \leq \beta < 1$, if and only if* $Re\left\{\dfrac{\partial_q (z\partial_q f(z))}{\partial_q f(z)}\right\} > \beta, \qquad$ *for all* $z \in \mathcal{U}$.

We denote by $K_q(\beta)$ the subclass of $\mathcal{A}$ consisting of all convex functions of order β in the unit disk $\mathcal{U}$.

We note that $f \in K_q(\beta)$ if and only if $z\partial_q f \in S_q^*(\beta)$

and $\lim\limits_{q \to 1^-} S_q^*(\beta) = S^*(\beta)$, and $\lim\limits_{q \to 1^-} K_q(\beta) = K(\beta)$,

where $S^*(\beta)$, $K(\beta)$ are the classes of starlike and convex functions of order β respectively.

2. Main results

Theorem 2.1. *A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is in $S_q^*(\beta)$, if*

$$\sum_{k=2}^{\infty} ([k]_q - \beta) | a_k | \leq 1 - \beta. \tag{2.1}$$

Proof: It suffices to show that the values for $\dfrac{z\partial_q f(z)}{f(z)}$ lie in a circle centered at $\omega = 1$ whose radius is $1-\beta.$

$$\left|\frac{z\partial_q f(z)}{f(z)}-1\right| = \left|\frac{z\partial_q f(z)-f(z)}{f(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}([k]_q-1)a_k z^k}{z+\sum_{k=2}^{\infty}a_k z^k}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty}([k]_q-1)\,|a_k|\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}|a_k|\,|z|^{k-1}} \quad \leq \frac{\sum_{k=2}^{\infty}([k]_q-1)\,|a_k|}{1-\sum_{k=2}^{\infty}|a_k|}.$$

This last expression is bounded above by $1-\beta$ if

$$\sum_{k=2}^{\infty}([k]_q-1)\,|a_k| \leq (1-\beta)\left(1-\sum_{k=2}^{\infty}|a_k|\right),$$

which is equivalent to

$$\sum_{k=2}^{\infty}([k]_q-\beta)\,|a_k| \leq 1-\beta. \tag{2.2}$$

But (2.2) is true by hypothesis. Hence $\left|\dfrac{z\partial_q f(z)}{f(z)}-1\right| \leq 1-\beta.$ This completes the proof.

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.2. *Let* $f(z) = z+\sum_{k=2}^{\infty}a_k z^k$. *if* $\sum_{k=2}^{\infty}(k-\beta)\,|a_k| \leq 1-\beta,$ *then* $f \in S^*(\beta)$.

As $q \to 1$ and $\beta = 0$, we have following result, proved by Goodman [1].

Corollary 2.3. *Let* $f(z) = z+\sum_{k=2}^{\infty}a_k z^k$. *if* $\sum_{k=2}^{\infty}k\,|a_k| \leq 1,$ *then* $f \in S^*$.

Also special case of Theorem 2.1 when $q \to 1$ and $\beta = \dfrac{1}{2}$ proved by Schild [3].

Theorem 2.4. *A function* $f(z) = z+\sum_{k=2}^{\infty}a_k z^k$ *is in* $C_q(\beta)$, *if*

$$\sum_{k=2}^{\infty}[k]_q([k]_q-\beta)\,|a_k| \leq 1-\beta. \tag{2.3}$$

Proof: It can be easily seen that $f(z) \in C_q(\beta)$ if and only if $z\partial_q f(z) \in S_q^*(\beta)$. Since $z\partial_q f(z) = z+\sum_{k=2}^{\infty}[k]_q a_k z^k$, we may replace a_k with $[k]_q a_k$ in the theorem.

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.5. *A function* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *is in* $C(\beta)$, *if* $\sum_{k=2}^{\infty} k(k-\beta)|a_k| \le 1-\beta.$

Theorem 2.6. *A function* $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$ *is in* $T_q^*(\beta)$, *if and only if*

$$\sum_{k=2}^{\infty}([k]_q - \beta)|a_k| \le 1-\beta. \tag{2.4}$$

Proof: In view of Theorem 2.1, it suffices to show the only if part. Assume that

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} = Re\left\{\frac{z - \sum_{k=2}^{\infty}[k]_q |a_k| z^k}{z - \sum_{k=2}^{\infty}|a_k| z^k}\right\} > \beta, \qquad |z|<1. \tag{2.5}$$

Choose values of z on the real axis so that $\frac{z\partial_q f(z)}{f(z)}$ is real. Upon clearing the denominator in (2.5) and letting $z \to 1$ through real values, we obtain

$$1 - \sum_{k=2}^{\infty}[k]_q |a_k| \ge \beta\left(1 - \sum_{k=2}^{\infty}|a_k|\right).$$

Thus $\sum_{k=2}^{\infty}([k]_q - \beta)|a_k| \le 1-\beta,$ and this completes the proof.

As $q \to 1$, we have following result proved by Herb silverman [5].

Corollary 2.7. *A function* $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$ *is in* $T^*(\beta)$, *if and only if* $\sum_{k=2}^{\infty}(k-\beta)|a_k| \le 1-\beta.$

Corollary 2.8. *If* $f \in T_q^*(\beta)$ *then* $|a_k| \le \frac{1-\beta}{[k]_q - \beta}$, *with equality only for functions of the form* $f_k(z) = z - \frac{(1-\beta)z^k}{[k]_q - \beta}$.

Theorem 2.9. *A function* $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$ *is in* $C_q(\beta)$, *if and only if*

$$\sum_{k=2}^{\infty}[k]_q([k]_q - \beta)|a_k| \le 1-\beta. \tag{2.6}$$

Proof: The proof follows as that of the Theorem 2.4.

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.10. *A function* $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k$ *is in* $C(\beta)$*, if and only if* $\sum_{k=2}^{\infty}k(k-\beta)|a_k|\le 1-\beta.$

Theorem 2.11. *If* $f\in T$*, then* $\sum_{k=2}^{\infty}[k]_q|a_k|\le 1.$

Proof: Suppose $\sum_{k=2}^{\infty}[k]_q|a_k|=1+\varepsilon,(\varepsilon>0).$ Then there exists an integer N such that $\sum_{k=2}^{N}[k]_q|a_k|>1+\frac{\varepsilon}{2}$. For z in the interval $\left(\frac{1}{1+\frac{\varepsilon}{2}}\right)^{\frac{1}{N-1}}<z<1,$ we have

$$\partial_q f(z)\le 1-\sum_{k=2}^{N}[k]_q|a_k|z^{k-1}\le 1-z^{N-1}\sum_{k=2}^{N}[k]_q|a_k|<1-\left(1+\frac{\varepsilon}{2}\right)z^{N-1}<0.$$

since $\partial_q f(0)>0,$ there exists a real number z_0, $0<z_0<1,$ for which $\partial_q f(z_0)=0.$ Hence $f(z)\notin T,$ and the theorem is proved.

As $q\to 1$, we have following result proved by Herb silverman [5].

Corollary 2.12. *If* $f\in T$*, then* $\sum_{k=2}^{\infty}k|a_k|\le 1.$

Theorem 2.13. *If* $f\in T_q^*(\beta),$ *then*

$$r-\frac{1-\beta}{[2]_q-\beta}r^2\le|f(z)|\le r+\frac{1-\beta}{[2]_q-\beta}r^2 \qquad (|z|=r),$$

with equality for $f(z)=z-\frac{(1-\beta)z^2}{[2]_q-\beta}$, $\quad(z=\pm r).$

Proof: Note that $([2]_q-\beta)\sum_{k=2}^{\infty}|a_k|\le\sum_{k=2}^{\infty}([k]_q-\beta)|a_k|\le 1-\beta,$ this last inequality following from Theorem 2.6. Thus

$$|f(z)|\le r+\sum_{k=2}^{\infty}|a_k|r^k\le r+r^2\sum_{k=2}^{\infty}|a_k|\le r+\frac{1-\beta}{[2]_q-\beta}r^2.$$

Similarly,

$$|f(z)|\ge r-\sum_{k=2}^{\infty}|a_k|r^k\ge r-r^2\sum_{k=2}^{\infty}|a_k|\ge r-\frac{1-\beta}{[2]_q-\beta}r^2.$$

As $q\to 1$, we have following result proved by Herb silverman[5].

Corollary 2.14. *If* $f\in T^*(\beta),$ *then*

$$r-\frac{1-\beta}{2-\beta}r^2 \le |f(z)| \le r+\frac{1-\beta}{2-\beta}r^2 \qquad (|z|=r),$$

with equality for $f(z)=z-\frac{(1-\beta)z^2}{2-\beta}, \quad (z=\pm r).$

Theorem 2.15. *If* $f \in C_q(\beta)$, *then*

$$r-\frac{1-\beta}{[2]_q([2]_q-\beta)}r^2 \le |f(z)| \le r+\frac{1-\beta}{[2]_q([2]_q-\beta)}r^2 \qquad (|z|=r),$$

with equality for $f(z)=z-\frac{(1-\beta)z^2}{[2]_q([2]_q-\beta)}, \quad (z=\pm r).$

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.16. *If* $f \in C(\beta)$, *then*

$$r-\frac{1-\beta}{2(2-\beta)}r^2 \le |f(z)| \le r+\frac{1-\beta}{2(2-\beta)}r^2 \qquad (|z|=r),$$

with equality for $f(z)=z-\frac{(1-\beta)z^2}{2(2-\beta)}, \quad (z=\pm r).$

Theorem 2.17. *The disk* $|z|<1$ *is mapped onto a domain that contains the disk* $|\omega|<\frac{[2]_q-1}{[2]_q-\beta}$ *by any* $f \in T_q^*(\beta)$, *and onto a domain that contains the disk* $|\omega|<\frac{([2]_q-1)\left|[2]_q+1)-\beta\right|}{[2]_q([2]_q-\beta)}$, *by any* $f \in C_q(\beta)$. *The theorem is sharp, with extremal functions* $f(z)=z-\frac{(1-\beta)z^2}{[2]_q-\beta} \in T_q^*(\beta)$ *and* $f(z)=z-\frac{(1-\beta)z^2}{[2]_q([2]_q-\beta)} \in C_q(\beta)$.

Proof: The results follow upon letting $r \to 1$, in Theorem 2.13 and Theorem 2.15.

As $q \to 1$, we get Theorem 5 in [5].

Theorem 2.18. *If* $f \in T_q^*(\beta)$, *then*

$$1-\frac{[2]_q(1-\beta)}{[2]_q-\beta}r \le |\partial_q f(z)| \le 1+\frac{[2]_q(1-\beta)}{[2]_q-\beta}r \qquad (|z|=r).$$

Equality holds for $f(z)=z-\frac{(1-\beta)z^2}{[2]_q-\beta}, \quad (z=\pm r).$

Proof: We have

$$|\partial_q f(z)| \le 1 + \sum_{k=2}^{\infty} [k]_q |a_k| |z|^{k-1} \le 1 + r \sum_{k=2}^{\infty} [k]_q |a_k|. \tag{2.7}$$

From Theorem 2.6, we have

$$\sum_{k=2}^{\infty} [k]_q |a_k| \le 1 - \beta + \beta \sum_{k=2}^{\infty} |a_k| \le 1 - \beta + \frac{\beta(1-\beta))}{[2]_q - \beta} = \frac{[2]_q (1-\beta)}{[2]_q - \beta}. \tag{2.8}$$

A substitution of (2.8) into (2.7) we obtained $|\partial_q f(z)| \le 1 + \dfrac{[2]_q (1-\beta)}{[2]_q - \beta} r$. On the other hand,

$$|\partial_q f(z)| \ge 1 - \sum_{k=2}^{\infty} [k]_q |a_k| |z|^{k-1} \ge 1 - r \sum_{k=2}^{\infty} [k]_q |a_k| \ge 1 - \frac{[2]_q (1-\beta)}{[2]_q - \beta} r,$$

and the proof is complete.

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.19. *If* $f \in T(\beta)$, *then*

$$1 - \frac{2(1-\beta)}{2-\beta} r \le |f'(z)| \le 1 + \frac{2(1-\beta)}{2-\beta} r \qquad (|z| = r).$$

Equality holds for $f(z) = z - \dfrac{(1-\beta)z^2}{2-\beta}, \quad (z = \pm r).$

Theorem 2.20. *If* $f \in C_q(\beta)$, *then*

$$1 - \frac{(1-\beta)}{[2]_q - \beta} r \le |\partial_q f(z)| \le 1 + \frac{(1-\beta)}{[2]_q - \beta} r \qquad (|z| = r).$$

Equality holds for $f(z) = z - \dfrac{(1-\beta)z^2}{[2]_q([2]_q - \beta)}, \quad (z = \pm r).$

Theorem 2.21. *If* $f \in T_q^*(\beta)$, *then* f *is* q*-convex in the disk*

$$|z| < r(\beta) = \inf_k \left(\frac{[k]_q - \beta}{[k]_q^2 (1-\beta)} \right)^{\frac{1}{k-1}} \qquad (n = 2,3,...).$$

The result is sharp, with the extremal function being of the form $f_k(z) = z - \dfrac{(1-\beta)z^k}{(k-\beta)}$ for some n.

Proof: It suffices to show that $\left| \dfrac{z\partial_q^2 f(z)}{\partial_q f(z)} \right| \le 1$ for $|z| \le r(\beta)$. We have

$$\left|\frac{z\partial_q^2 f(z)}{\partial_q f(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}[k]_q([k]_q-1)|a_k|z^{k-1}}{1+\sum_{k=2}^{\infty}[k]_q|a_k|z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}[k]_q([k]_q-1)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}[k]_q|a_k||z|^{k-1}}.$$

Thus $\left|\frac{z\partial_q^2 f(z)}{\partial_q f(z)}\right| \le 1$ if $\sum_{k=2}^{\infty}[k]_q([k]_q-1)|a_k||z|^{k-1} \le 1-\sum_{k=2}^{\infty}[k]_q|a_k||z|^{k-1}$,

or

$$\sum_{k=2}^{\infty}[k]_q^2|a_k||z|^{k-1} \le 1. \tag{2.9}$$

According to Theorem 2.6, $\frac{\sum_{k=2}^{\infty}([k]_q-\beta)|a_k|}{(1-\beta)} \le 1$. Hence (2.9) will be true if

$$[k]_q^2|z|^{k-1} \le \frac{[k]_q-\beta}{1-\beta} \quad (n=2,3,...). \tag{2.10}$$

Solving (2.10) for z, we obtain

$$|z| \le \left(\frac{[k]_q-\beta}{[k]_q^2(1-\beta)}\right)^{\frac{1}{k-1}} \quad (n=2,3,...). \tag{2.11}$$

Sitting $|z| = r(\beta)$ in (2.11), the result is follows.

Theorem 2.22. *Let* $f_1(z) = z$ *and* $f_k(z) = z - \frac{1-\beta}{[k]_q-\beta}z^k$ $(k=2,3,...)$.

Then $f \in T_q^*(\beta)$ *if and only if it can expressed in the form*

$$f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z), \quad \text{where } \lambda_k > 0 \text{ and } \sum_{k=1}^{\infty}\lambda_k = 1.$$

Proof: Suppose that

$$f(z) = \sum_{k=1}^{\infty}\lambda_k f_k(z) = f_1(z)+\sum_{k=2}^{\infty}\lambda_k f_k(z) = \lambda_1 z+\sum_{k=2}^{\infty}\lambda_k\left(z-\frac{1-\beta}{[k]_q-\beta}z^k\right)$$

$$= \lambda_1 z+\sum_{k=2}^{\infty}\lambda_k z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{[k]_q-\beta}z^k = \left(\sum_{k=1}^{\infty}\lambda_k\right)z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{[k]_q-\beta}z^k$$

$$= z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{[k]_q-\beta}z^k.$$

Then

$$\sum_{k=2}^{\infty}\lambda_k\left(\frac{1-\beta}{[k]_q-\beta}\right)\left(\frac{[k]_q-\beta}{1-\beta}\right) = \sum_{k=2}^{\infty}\lambda_k = 1-\lambda_1 \le 1.$$

Thus $f \in T_q^*(\beta)$.

Conversely, suppose $f \in T_q^*(\beta)$. Since $|a_k| \le \dfrac{1-\beta}{[k]_q - \beta}$ $(k = 2,3,...)$, we may set $\lambda_k = \dfrac{[k]_q - \beta}{1-\beta}$ and $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$. Then

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

And the proof of Theorem 2.22 is complete.

As $q \to 1$, we have following result proved by Herb silverman[5].

Corollary 2.23. *Let* $f_1(z) = z$ *and* $f_k(z) = z - \dfrac{1-\beta}{k-\beta} z^k$ $(k = 2,3,...)$. *Then*

$f \in T^*(\beta)$ *if and only if it can expressed in the form* $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$,

where $\lambda_k > 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

3. Partial sums

In this section, we will examine the ratio of a function of the form (1.1) to its sequence of partial sums $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$ when the coefficients of f are sufficiently small to satisfy the condition (2.4). We will determine sharp lower bounds for

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\}, \Re\left\{\frac{f_k(z)}{f(z)}\right\}, \Re\left\{\frac{\partial_q f(z)}{\partial_q f_k(z)}\right\} \text{ and } \Re\left\{\frac{\partial_q f_k(z)}{\partial_q f(z)}\right\}.$$

Theorem 3.1. *If* f *of the form (1.1) and satisfies condition (2.4) , then*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} \ge 1 - \frac{1}{c_{k+1}}, \quad (z \in \mathrm{U}, k \in \mathrm{N}), \tag{3.1}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \ge \frac{c_{k+1}}{1 + c_{k+1}}, \quad (z \in \mathrm{U}, k \in \mathrm{N}), \tag{3.2}$$

where $c_k = \dfrac{([k]_q - \beta)}{1-\beta}$. The estimates in (3.1) and (3.2) are sharp.

Proof: Suppose that f satisfies condition (2.4), by Theorem 2.6, we have

$$f \in T_q^*(\beta) \Leftrightarrow \sum_{n=2}^{\infty} c_n |a_n| \le 1,$$

It is easy to verify that $c_{n+1} > c_n > 1$. Thus,

$$\sum_{n=2}^{k}|a_n|+c_{k+1}\sum_{n=k+1}^{\infty}|a_n|\leq\sum_{n=2}^{\infty}c_n|a_n|\leq 1. \tag{3.3}$$

We may write

$$c_{k+1}\left\{\frac{f(z)}{f_k(z)}-\left(1-\frac{1}{c_{k+1}}\right)\right\}=\frac{1+\sum_{n=2}^{k}a_nz^{n-1}+c_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{k}a_nz^{n-1}}=\frac{1+A(z)}{1+B(z)}.$$

Set

$$\frac{1+A(z)}{1+B(z)}=\frac{1+w(z)}{1-w(z)},$$

so that $w(z)=\dfrac{B(z)-A(z)}{2+B(z)+A(z)}$, then $w(z)=\dfrac{c_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}{2+2\sum_{n=2}^{k}a_nz^{n-1}+c_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}$,

and
$$|w(z)|\leq\frac{c_{k+1}\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{k}|a_n|-c_{k+1}\sum_{n=k+1}^{\infty}|a_n|}.$$

Now $|w(z)|\leq 1$ if and only if $\sum_{n=2}^{k}|a_n|+c_{k+1}\sum_{n=k+1}^{\infty}|a_n|\leq 1$,

which is true by (3.3). This readily yields the assertion (3.1).

To see that

$$f(z)=z-\frac{z^{k+1}}{c_{k+1}}, \tag{3.4}$$

gives sharp results, we observe that for $z=re^{\frac{\pi i}{k}}$

$$\frac{f(z)}{f_k(z)}=1+\frac{z^k}{c_{k+1}}.$$

Letting $z\to 1^-$, we have $\dfrac{f(z)}{f_k(z)}=1-\dfrac{1}{c_{k+1}}$,

which shows that the bounds in (3.1) are the best possible for each $n\in\mathbb{N}$.

In the same way we take

$$(1+c_{k+1})\left(\frac{f_k(z)}{f(z)}-\frac{c_{k+1}}{1+c_{k+1}}\right)=\frac{1+\sum_{n=2}^{\infty}a_nz^{n-1}+c_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}=\frac{1+w(z)}{1-w(z)},$$

where $$|w(z)| \le \frac{1+c_{k+1}\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{\infty}|a_n|-(1+c_{k+1})\sum_{n=k+1}^{\infty}|a_n|}.$$

Now $|w(z)| \le 1$ if and only if $\sum_{n=2}^{k}|a_n|+(1+c_{k+1})\sum_{n=k+1}^{\infty}|a_n| \le 1$,
which is true by (3.3). This readily yields the assertion (3.2).

The estimate in (3.2) is sharp with the extremal function $f(z)$ given by (3.4). This completes the proof of Theorem.

Theorem 3.2. *If* f *If* f *of the form (1.1) and satisfies condition (2.4), then*

$$\Re\left\{\frac{\partial_q f(z)}{\partial_q f_k(z)}\right\} \ge 1-\frac{[k]_q+1}{c_{k+1}}, \quad (z\in \mathrm{U}, k\in \mathrm{N}), \tag{3.5}$$

and

$$\Re\left\{\frac{\partial_q f_k(z)}{\partial_q f(z)}\right\} \ge \frac{c_{k+1}}{[k]_q+1+c_{k+1}}, \quad (z\in \mathrm{U}, k\in \mathrm{N}). \tag{3.6}$$

where $c_k = \dfrac{([k]_q-\beta)}{1-\beta}$. The estimates in (3.5) and (3.6) are sharp with the extremal function given by (3.4).

Proof: We may write

$$c_{k+1}\left\{\frac{\partial_q f(z)}{\partial_q f_k(z)}-\left(1-\frac{[k]_q+1}{c_{k+1}}\right)\right\} = \frac{1+\sum_{n=2}^{k}[n]_q a_n z^{n-1}+\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q a_n z^{n-1}}{1+\sum_{n=2}^{k}[n]_q a_n z^{n-1}} = \frac{1+A(z)}{1+B(z)}.$$

Set $\dfrac{1+A(z)}{1+B(z)} = \dfrac{1+w(z)}{1-w(z)}$, so that $w(z) = \dfrac{B(z)-A(z)}{2+B(z)+A(z)}$,

then $$w(z) = \frac{-\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q a_n z^{n-1}}{2+2\sum_{n=2}^{k}[n]_q a_n z^{n-1}+\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q a_n z^{n-1}},$$

and

$$|w(z)| \le \frac{\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q |a_n|}{2-2\sum_{n=2}^{k}[n]_q |a_n|-\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q |a_n|},$$

Now $|w(z)|\leq 1$ if and only if $\dfrac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q|a_n|+\sum_{n=2}^{k}[n]_q|a_n|\leq 1.$

From the condition (2.4), it is suffices to show that

$$\frac{c_{k+1}}{[k]_q+1}\sum_{n=k+1}^{\infty}[n]_q|a_n|+\sum_{n=2}^{k}[n]_q|a_n|\leq c_n|a_n|.$$

This is equivalent to showing that

$$\sum_{n=2}^{k}(c_n-[n]_q)|a_n|+\sum_{n=k+1}^{\infty}\frac{([k]_q+1)c_n-nc_{k+1}}{[k]_q+1}\geq 0.$$

To prove the second part of this theorem, we write

$$w(z)=([k]_q+1+c_{k+1})\left\{\frac{\partial_q f(z)}{\partial_q f_k(z)}-\left(1-\frac{[k]_q+1}{c_{k+1}}\right)\right\}=1-\frac{\left(1+\frac{c_{k+1}}{[k]_q+1}\right)\sum_{n=k+1}^{\infty}[n]_q a_n z^{n-1}}{1+\sum_{n=2}^{k}[n]_q a_n z^{n-1}}$$

yields $\left|\dfrac{w(z)-1}{w(z)+1}\right|\leq\dfrac{\left(1+\frac{c_{k+1}}{[k]_q+1}\right)\sum_{n=k+1}^{\infty}[n]_q|a_n|}{2-2\sum_{n=2}^{k}[n]_q|a_n|-\left(1+\frac{c_{k+1}}{[k]_q+1}\right)\sum_{n=k+1}^{\infty}[n]_q|a_n|}\leq 1,(z\in\mathcal{U}),$

if and only if $2\left(1+\dfrac{c_{k+1}}{[k]_q+1}\right)\sum_{n=k+1}^{\infty}[n]_q|a_n|\leq 2-2\sum_{n=2}^{k}[n]_q|a_n|.$

The bound in (3.6)is sharp for all $n\in\mathbb{N}$ with the extremal function (3.4). This completes the proof.

REFERENCES

1. A.W.Goodman, Univalent functions and nonanalytic curves, *Poroc. Amer. Math. Soc.*, 8(3)(1957) 598-601.
2. F.H.Jackson, On q-functions and a certain difference operator, *Trans. Royal Soc. Edinburgh*, 46 (1909) 253-281.
3. A.Schild, On a class of univalent, star shaped mappings, *Poroc. Amer. Math. Soc.*, 9(5) (1958) 751-757.
4. T.M.Seoudy and M.K.Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, *J. Math. Inequal*, 10(1) (2016) 135-145.
5. H.Silverman, Univalent functions with negative coefficients, *Poroc. Amer. Math. Soc.*, 51(1) (1975) 109-116.
6. H.Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, 209(1) (1997) 221-227.
7. H.Shamsan, F.Alsarari and S.Latha, On coefficients inequalities of functions related to *q*-derivative and conic regions, *J.Math. Informatics*, 9 (2017) 9-17.