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All the solutions of the Diophantine Equation $p^x + (p+4)^y = z^2$ when p, $(p + 4)$ are Primes and $x + y = 2, 3, 4$

Nechemia Burshtein

117 Arlozorov Street, Tel Aviv 6209814, Israel
Email: anb17@netvision.net.il

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Abstract. In this paper we consider the Diophantine equation $p^x + (p+4)^y = z^2$ when $p > 2$, $(p + 4)$ are primes, and x, y, z are positive integers. All the possibilities of $x + y = 2, 3, 4$ are examined, and it is established that the equation has the unique solution $(p, x, y, z) = (3, 2, 1, 4)$.

Keywords: Diophantine equations, Cousin primes

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1. Introduction

A prime gap is the difference between two consecutive primes. Numerous articles have been written on prime gaps, a very minute fraction of which is brought [3,4] here. In 1849, A. de Polignac conjectured that for every positive integer k, there are infinitely many primes p such that $p + 2k$ is prime too. Many questions and conjectures on the above still remain unanswered and unsolved.

When $k = 1$, the pairs $(p, p + 2)$ are known as Twin Primes. The first four such pairs are: (3, 5), (5, 7), (11, 13), (17, 19). The Twin Prime conjecture stating that there are infinitely many such pairs remains unproved. When $k = 2$, the pairs $(p, p + 4)$ are called Cousin Primes. The first six pairs are: (3, 7), (7, 11), (13, 17), (19, 23), (37, 41), (43, 47).

In this paper, the known Diophantine equation $p^x + q^y = z^2$ [see 1, 5, 6, 7] is considered when p and q are Cousin Primes i.e.,

$$p^x + (p+4)^y = z^2, \qquad (1)$$

and x, y, z are positive integers. We examine all the possibilities of $x + y = 2, 3, 4$ for solutions of equation (1). This is done in Section 2.

2. The equation $p^x + (p+4)^y = z^2$

In this section we prove the following result.

Theorem 2.1. Suppose that $p > 2$, $(p + 4)$ are any two primes, and x, y, z are positive integers. If $x + y = 2, 3, 4$, then the equation $p^x + (p + 4)^y = z^2$ has the unique solution

$$3^2 + 7^1 = 4^2.$$

Proof: For $x + y = 2, 3, 4$, we examine all possible values x, y. These are:

Case 1.	$x + y = 2$	$x = 1,$	$y = 1.$
Case 2.	$x + y = 3$	$x = 1,$	$y = 2.$
Case 3.	$x + y = 3$	$x = 2,$	$y = 1.$
Case 4.	$x + y = 4$	$x = 1,$	$y = 3.$
Case 5.	$x + y = 4$	$x = 2,$	$y = 2.$
Case 6.	$x + y = 4$	$x = 3,$	$y = 1.$

Each of these cases is considered separately. The value z^2 is even, hence z is even. Thus z^2 is a multiple of 4.

Case 1. Suppose in equation (1) $x = 1$ and $y = 1$. We then obtain $p + (p + 4) = z^2$ or

$$2(p + 2) = z^2. \tag{2}$$

But, the left-hand side of (2) is a multiple of 2 only, whereas the right-hand of (2) is a multiple of 4. Since this is impossible, equation (1) has no solution in this case.

Case 2. Suppose in equation (1) $x = 1$ and $y = 2$. We have

$$p^1 + (p + 4)^2 = z^2$$

implying $p^2 + 9p + 16 = z^2$ or

$$p(p + 9) = (z - 4)(z + 4). \tag{3}$$

From (3) it follows that p divides at least one of the values $z - 4$, $z + 4$.
If $p \mid (z - 4)$, denote $pA = z - 4$ where A is an even integer. Thus, $p(p + 9) = pA(pA + 8)$ implying

$$p + 9 = A(pA + 8). \tag{4}$$

For any prime p, (4) clearly implies that all values A are impossible. Thus, (4) does not exist, and $p \nmid (z - 4)$.
If $p \mid (z + 4)$, denote $pB = z + 4$ where B is an even integer. Then from (3) we have $p(p + 9) = (pB - 8)\, pB$ or $p + 9 = B(pB - 8)$ which yields

$$p = \frac{9 + 8B}{B^2 - 1}. \tag{5}$$

Consequently, one can see that the right-hand side of (5) is never equal to an integer p implying that (5) is impossible, and $p \nmid (z + 4)$.

Hence, Case 2 does not yield a solution of equation (1).

Case 3. Suppose in equation (1) $x = 2$ and $y = 1$. We have

$$p^2 + (p + 4)^1 = z^2$$

and

$$p(p + 1) = z^2 - 4 = (z - 2)(z + 2). \tag{6}$$

Thus, from (6), p divides at least one of the values $z - 2$, $z + 2$.
If $p \mid (z - 2)$, denote $pC = z - 2$ and $pC + 4 = z + 2$ where C is an even integer. From (6) we then have

$$p + 1 = C(pC + 4)$$

which is clearly impossible for all values C.
If $p \mid (z + 2)$, denote $pD = z + 2$ and $pD - 4 = z - 2$ where D is an even integer. From (6) we have

$$p + 1 = (pD - 4)D \tag{7}$$

which yields the two smallest possible values $p = 3$ and $D = 2$ as a solution of (7). Hence

$$3^2 + 7 = 4^2$$

is a solution of equation (1).

Case 4. Suppose in equation (1) $x = 1$ and $y = 3$. We obtain

$$p^1 + (p + 4)^3 = p + (p^3 + 12p^2 + 48p + 64) = z^2.$$

Thus

$$p + p^3 + 4(3p^2 + 12p + 16) = z^2. \tag{8}$$

Since z^2 is a multiple of 4, it follows from (8) that $1 + p^2$ must be a multiple of 4. Every prime $p \geq 3$ is of the form $4N + 3$ $(N \geq 0)$ or $4N + 1$ $(N \geq 1)$. It is then easily seen in either case, that the value $1 + p^2$ is never a multiple of 4.
Hence, Case 4 is impossible, and does not contribute a solution to equation (1).

Case 5. Suppose in equation (1) $x = 2$ and $y = 2$. We have

$$p^2 + (p + 4)^2 = p^2 + (p^2 + 8p + 16) = z^2,$$

hence

$$2p^2 + (8p + 16) = z^2. \tag{9}$$

But, z^2 and $(8p + 16)$ are multiples of 4, whereas $2p^2$ is not. Thus, equality (9) is impossible, implying that equation (1) has no solutions in this case.

Case 6. Suppose in equation (1) $x = 3$ and $y = 1$. We have

$$p^3 + (p + 4)^1 = z^2. \tag{10}$$

As in Case 4, the value $p^3 + p$ must be a multiple of 4 since 4 and z^2 are multiples of 4. Using the argument in Case 4, it follows that $4 \nmid (p^3 + p)$. Equality (10) is therefore impossible, and no solution of equation (1) exists in this case.

Thus, $3^2 + 7^1 = 4^2$ is the unique solution as asserted.

The proof of Theorem 2.1 is complete. □

3. Conclusion

In this paper, we have established for any two primes $p > 2$, $(p + 4)$, and positive integers x, y, z where $x + y = 2, 3, 4$, that the equation $p^x + (p+4)^y = z^2$ has a unique solution $(p, x, y, z) = (3, 2, 1, 4)$. The following question may now be raised.

Question 1. Let $p \geq 3$, $(p + 4)$ be any two primes, and x, y, z are positive integers. If x, y satisfy $x + y > 4$, does the equation $p^x + (p+4)^y = z^2$ have solutions ?

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