Annals of Pure and Applied Mathematics
Vol. 16, No. 1, 2018, 31-35
ISSN: 2279-087X (P), 2279-0888(online)
Published on 1 January 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v16n1a4

Annals of
Pure and Applied Mathematics

All the Solutions to an Open Problem of S. Chotchaisthit on the Diophantine Equation $2^x + p^y = z^2$ when p are Particular Primes and $y = 1$

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Received 11 December 2017; accepted 14 December 2017

Abstract. An Open Problem of S. Chotchaisthit in [5] states that finding solutions of the title equation where p is prime in general is still an open problem. Moreover, it is not known how to find all non-negative integer solutions of the equation for the particular primes $p = 7, 13, 29, 37, 257$. In this paper, among other results on the equation, we establish all the solutions in positive integers of the equation when $y = 1$ and $p = 7, 13, 29, 37, 257$. All the solutions are exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 2, 5, 6, 7, 9].

Chotchaisthit [5] considers the equation $2^x + 11^y = z^2$, stating that it is not known how to find all non-negative integer solutions of $2^x + p^y = z^2$ when $p = 7, 13, 29, 37, 257$.

In this paper, we obtain some results for the equation

$$2^x + p^y = z^2 \qquad p \text{ prime} \qquad y = 1. \tag{1}$$

When $p = 7, 13, 29, 37, 257$, all the solutions in positive integers are established and exhibited.

2. Some results on $2^x + p = z^2$

Lemma 2.1. Suppose that $2^x + p = z^2$ where $x \ge 2$. If $p = 4N + 3$ ($N \ge 0$), then the equation has no solutions.

Proof: We shall assume that for a certain prime $p_s = 4N + 3$ the equation has a solution, and reach a contradiction.

For each and every prime $p \ge 3$, the value z is odd. Denote $z = 2M + 1$. Hence, $z^2 = (2M + 1)^2 = 4M^2 + 4M + 1$. By our assumption we have

$$z^2 = 2^x + p_s = 2^x + (4N + 3) = 2^x + (4N + 2) + 1 = 4M^2 + 4M + 1$$

where $2^x + 4N + 2 = 2(2^{x-1} + 2N + 1)$ and

$$2(2^{x-1} + 2N + 1) = 4M(M + 1).$$

But, the above equality is false. This is easily seen, since the right-hand side is at least a multiple of 8, whereas the left-hand side is a multiple of 2 only.

Our assumption that there exists a prime $p_s = 4N + 3$ which yields a solution to the equation is therefore false.

This completes the proof of Lemma 2.1. □

However, for the value $x = 1$, there exist solutions of the equation when $p = 4N + 3$. The first four solutions with primes of this form are demonstrated in the following Table 1.

Table 1.

x	2^x	$p = 4N + 3$	z^2
1	2	7	9
1	2	23	25
1	2	47	49
1	2	79	81

Lemma 2.2. Suppose that $2^x + p = z^2$. If x is even, i.e., $x = 2t$, then 2^x, p and z are of the following form:

$(z - 1)^2 = 2^x$ or $2^x = 2^{2t} = (2^t)^2$,

$z - 1 = 2^t$ or $z = 2^t + 1$,

$2z - 1 = p$ or $p = 2^{t+1} + 1$.

Proof: The equation $2^x + p = z^2$ corresponds to the identity $(z - 1)^2 + (2z - 1) = z^2$. When $x = 2t$, the values 2^x, p and z follow immediately. □

Corollary 2.1. Suppose that $2^x + p = z^2$ where $x \ge 2$. If the equation has any solutions, then $p = 4N + 1$ ($N \ge 1$).

3. On the solutions of $2^x + p = z^2$ when $p = 7, 13, 29, 37, 257$

The case $p = 7$. In the following Lemma 3.1, we consider equation (1) when $p = 7$.

Lemma 3.1. If in equation (1) $p = 7$, then the equation has one and only one solution when $x = 1$ and $z = 3$.

Proof: When $p = 7$, then the following solution of equation (1) is

$$2^1 + 7 = 3^2. \tag{2}$$

We now show that this is the only possible solution.

Suppose that there exist values $x > 1$ and $z > 3$ which yield a solution of equation (1), namely

$$2^x + 7 = z^2. \tag{3}$$

Then, from (2) and (3), we have

$$2^x - 2 = z^2 - 3^2 = (z - 3)(z + 3) = 2(2^{x-1} - 1). \tag{4}$$

From (4) it follows that 2 divides at least one of the values $(z-3)$, $(z+3)$. If $2 \mid (z-3)$, then $2V = z-3$ or $z = 2V+3$ implying that $z + 3 = 2V + 6$. Therefore, $(z-3)(z+3) = (2V)(2V+6) = 4(V)(V+3)$. If $2 \mid (z+3)$, one obtains a result of the same nature. Hence, in (4) the value $2(2^{x-1}-1)$ is a multiple of 2 only, whereas $(z-3)(z+3)$ is at least a multiple of 4. Since this is impossible, it follows that our supposition (3) is false, and equality (2) is the only solution of equation (1) when $p = 7$.

Solution (2) is unique.

This completes our proof. □

The case $\boldsymbol{p = 13, 29, 37}$. In Lemma 3.2, we consider equation (1) when $p = 13, 29, 37$. The primes are of the form $4N + 1$ as in Corollary 2.1.

Lemma 3.2. Suppose that $2^x + p = z^2$. If $p = 13, 29, 37$, then the equation has no solutions.

Proof: When $x = 1$, it is clearly seen that the equation has no solution for each of these primes. Therefore, let $x > 1$. We shall distinguish two cases for x, namely: $x = 2t$ and $x = 2t + 1$ where $t \geq 1$.

Suppose that $x = 2t$. Then, from Lemma 2.2 we have:

$p = 13 = 2^{t+1} + 1$ or $2^{t+1} = 12$ is impossible.

$p = 29 = 2^{t+1} + 1$ or $2^{t+1} = 28$ is impossible.

$p = 37 = 2^{t+1} + 1$ or $2^{t+1} = 36$ is impossible.

Hence, when x is even, the equation has no solutions.

Suppose that $x = 2t + 1$. The odd value z is of the form $4T + 1$ or $4T + 3$. We shall consider the two possibilities separately.

Let $z = 4T + 1$. When $p = 13$ we have

$2^{2t+1} + 13 = (4T+1)^2 \quad 2^{2t+1} + 3 \cdot 4 + 1 = 16T^2 + 8T + 1 \quad 2^{2t+1} + 3 \cdot 4 = 16T^2 + 8T$

or

$$2^{2t-1} + 3 = 4T^2 + 2T \qquad \text{is impossible.}$$

When $p = 29, 37$, and in the same manner as for $p = 13$, we respectively obtain:

$2^{2t+1} + 29 = (4T+1)^2 \qquad 2^{2t-1} + 7 = 4T^2 + 2T$ is impossible.

$2^{2t+1} + 37 = (4T+1)^2 \qquad 2^{2t-1} + 9 = 4T^2 + 2T$ is impossible.

Hence $z \neq 4T + 1$.

Let $z = 4T + 3$. When $p = 13$ we have

$2^{2t+1} + 13 = (4T+3)^2 \qquad 2^{2t+1} + 3 \cdot 4 + 1 = 16T^2 + 24T + 9 = 8(2T^2 + 3T + 1) + 1$

$2^{2t+1} + 3 \cdot 4 = 8(2T^2 + 3T + 1) \qquad 2^{2t-1} + 3 = 2(2T^2 + 3T + 1)$ is impossible.

For $p = 29, 37$, and in the same manner as for $p = 13$, we respectively have:

$2^{2t+1} + 29 = (4T+3)^2 \qquad 2^{2t-1} + 7 = 2(2T^2 + 3T + 1)$ is impossible.

$2^{2t+1} + 37 = (4T+3)^2 \qquad 2^{2t-1} + 9 = 2(2T^2 + 3T + 1)$ is impossible.

Thus $z \neq 4T + 3$.

Therefore, when x is odd, no value z exists, and the equation has no solutions.

When $p = 13, 29, 37$, the equation has no solutions as asserted.

This concludes the proof of Lemma 3.2. □

The case $p = 257$. In the next lemma, we establish the only two solutions of the equation $2^x + 257 = z^2$.

Lemma 3.3. Suppose that $2^x + p = z^2$. If $p = 257$, then the equation has exactly two solutions, namely:

(i) $2^{14} + 257 = 129^2$,

(ii) $2^5 + 257 = 17^2$.

Proof: We shall consider separately the two cases $x = 2t$ and $x = 2t + 1$ where $t \geq 1$.

Let $x = 2t$. Then, by Lemma 2.2 it follows that $p = 2^{t+1} + 1$. The prime $p = 257 = 2^{t+1} + 1$ yields $t = 7$. Hence, $x = 2t = 14$, and $z = 129$. Thus, for all even values x

(i) $$2^{14} + 257 = 129^2$$

is the unique solution of equation (1).

Let $x = 2t + 1$. The equation $2^{2t+1} + 257 = z^2$ corresponds to the identity

$$2(2^t)^2 + (2^t)^4 + 1 = ((2^t)^2 + 1)^2.$$

Since $p = 257 = (2^t)^4 + 1$, it follows that $t = 2$, hence $x = 2t + 1 = 5$ and $z = 17$. Thus, for all odd values x

(ii) $$2^5 + 257 = 17^2$$

is the unique solution of equation (1).

We have shown that the equation $2^x + 257 = z^2$ has exactly two solutions. One solution when x is even, the other solution when x is odd.

The proof of Lemma 3.3 is complete. □

Remark 3.1. It is easily seen for the primes $p = 7, 13, 29, 37, 257$, that equation (1) has no solutions when $x = 0$.

4. Conclusion

In this paper we have considered positive integral solutions of the equation $2^x + p^y = z^2$ when $y = 1$. An emphasis has been made on the primes $p = 7, 13, 29, 37, 257$. It has been established: when $p = 7$, then $2 + 7 = 3^2$ is the unique solution, whereas for $p = 257$, the respective solutions $2^{14} + 257 = 129^2$ and $2^5 + 257 = 17^2$ are unique when x is even and when x is odd. For the primes, $p = 13, 29, 37$, it has been shown that the equation has no solutions.

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