Annals of Pure and Applied Mathematics
Vol. 16, No. 1, 2018, 69-80
ISSN: 2279-087X (P), 2279-0888(online)
Published on 1 January 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v16n1a9

Annals of
Pure and Applied Mathematics

Vertex Distance Complement Spectra of Some Graphs

Renny P Varghese[1] **and** ***Susha D.***[2]

Department of Mathematics, Catholicate College, Pathanamthitta
Kerala, India - 689645.
E-mail: [1]rennypv1@gmail.com, [2]sushad70@gmail.com

Received 6 December 2017; accepted 26 December 2017

Abstract. In this article we determine the *VDC*- spectrum of some class of graphs. We mainly discuss the *VDC* - spectrum of join of two graphs, Cartesian product, double graph, lexicographic product, double odd graph and extended double cover graph. We observe that under certain conditions some graphs in the above classification are *VDC*-integral. Also, we characterize that the Cartesian product of a graph G with K_2 and the extended double cover graph of G are *VDC*-equienergetic graphs.

Keywords: vertex distance complement matrix, join of graphs, Cartesian product, lexicographic product, double graph, double odd graph, extended double cover graph, *VDC* - energy.

AMS Mathematics Subject Classification (2010): 05C12, 05C50, 05C76

1. Introduction

Let G be a simple undirected graph on n vertices. The *adjacency matrix* of G denoted by $A(G) = (a_{ij})_{n\times n}$ is an $n \times n$ symmetric matrix indexed by the vertices $\{v_1, v_2, \ldots, v_n\}$ of G where $a_{ij} = 1$ if v_i and v_j are adjacent in G and is 0 otherwise. A graph is regular if every vertex has the same degree. The characteristic polynomial of G is defined as $f_G(x) = det\,(xI_n - A)$ where I_n is the identity matrix of order n. The roots of the characteristic equation of A are called the eigenvalues of G. It is denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and are called A - *spectrum* of G.

The distance matrix $D(G) = (d_{ij})$ where $d_{ij} = d(v_i, v_j)$ is the distance (the length of the shortest path) between the vertices v_i and v_j. If the diameter of G is atmost two, any pair of non adjacent vertices is at a distance less than or equal to two. *Graham* and *Pollak* [5] introduced the distance matrices in 1971. The matrix $D(G)$ is non negative, irreducible and symmetric, the eigenvalues of $D(G)$ are real. Let G be a graph with diameter atmost 2, then $D(G) = A(G) + 2\,\overline{A(G)} = 2(J - I) - A(G)$ [6] where $\bar{A}$ is the adjacency matrix of the complement graph $\bar{G}$. If $\eta_{i_1} \geq \eta_{i_2} \geq \cdots \geq \eta_{i_g}$ are the distinct eigenvalues of $D(G)$ with corresponding algebraic multiplicity $m_{i_1}, m_{i_2}, \ldots, m_{i_g}$ and $m_{i_1} + m_{i_2} + \ldots + m_{i_g} = n$, then the D - spectrum can be written as

$$Spec_D(G) = \begin{pmatrix} \eta_{i1} & \eta_{i2} & \eta_{ig} \\ m_{i1} & m_{i2} & m_{ig} \end{pmatrix}.$$

Now consider a special class of graphs called chemical graphs, which representing the chemical structure of a compound. Molecular graphs are graphs in which the chemical structure under consideration are molecules. Also, these molecular graphs are undirected graphs. In molecular graphs, vertices correspond to atoms and edge represents covalent bond between atoms and usually the hydrogen atoms are neglected. These matrices have been used to determine a number of topological indices like Balaban index, Winer index, distance sum index etc. There are some models for the molecular design of a chemical compound. Quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR) are two such models. *Renny* and *Susha* in [12] determined the *VDC* - spectrum of the molecular matrices derived from the graph distance namely *vertex distance complement matrix (VDC)*. In this paper we find the *VDC* - spectrum of some class of graphs namely join of two graphs, cartesian product, double graph, lexicographic product, double odd graph and extended double cover graph.

The organization of the paper is as follows. In section 2 we mention some basic results on spectral graph theory which are useful to prove the results in the succeeding sections. In section 3 we determine the *VDC* - spectrum of some class of graphs. We also discuss about some family of graphs which are *VDC* - integral. Then in section 4 we discuss the application of *VDC* - spectrum such as *VDC* - energy of $G \times K_2$, $G[K_2]$, $D_2(G)$ and the extended double cover graph. We conclude that $G \times K_2$ and extended double cover graph of G are *VDC* - equienergetic graphs.

2. Preliminaries

Definition 2.1. [8] The vertex distance complement matrix *VDC* = *VDC(G)* of a graph *G* with '*n*' vertices is an $n \times n$ symmetric matrix *VDC* = $[c_{ij}]$, where

$$c_{ij} = \begin{cases} n - d_{ij}, & \text{if } i \neq j \\ 0, & if\ i = j \end{cases}$$

d_{ij} is the distance between the vertices v_i and v_j.

The eigenvalues of *VDC*(*G*) are denoted by $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and are called the *VDC* - eigenvalue of *G*. The set of all *VDC* - eigenvalues of *G* is called the *VDC* - spectrum of *G*. Two non isomorphic graphs are said to be *VDC - cospectral* if they have the same *VDC* - spectrum. A graph is *VDC - integral* if the *VDC* - spectrum consists only of integers.

Definition 2.2. [3] Let G be a simple connected graph. G is called distance regular if it is regular, and if for any two vertices $u, v \in V(G)$ at a distance *i*, there are constant number of neighbors c_i and b_i of *v* at a distance $i - 1$ and $i + 1$ from *u* respectively.

Theorem 2.1. [12] Let *G* be a *r* - regular graph with *n* vertices and $diam(G) = 2$. Let $\{r, \lambda_2, \ldots, \lambda_n\}$ be the adjacency eigenvalues of *G*, then *VDC* - eigenvalues of *G* are $(n-1)(n-2) + r$ and $\lambda_i - n + 2$ for $i = 2, 3, \ldots, n$.

Lemma 2.2. [4] Let *G* be a connected *r* - regular graph on '*n*' vertices with its adjacency matrix A having *n* distinct eigenvalues $r = \lambda_1, \lambda_2, \ldots, \lambda_n$. Then there exists a polynomial $P(x) = n \frac{(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_n)}{(r-\lambda_2)(r-\lambda_3)\ldots(r-\lambda_n)}$ such that P (A) = J where J is the square matrix of order *n* whose all entries are one, so that P $(r) = n$ and P $(\lambda_i) = 0$ for all $\lambda_i \neq r$.

Lemma 2.3. [4] Let G be a connected r - regular graph with adjacency matrix A and spectrum $\{ r = \lambda_1, \lambda_2, \ldots, \lambda_n \}$. Then the adjacency matrix and spectrum of $\bar{G}$, the complement of the graph G, are $\bar{A}$ = J – I - A and $\{n - r - 1, -(\lambda_2 + 1), \ldots, -(\lambda_n + 1)\}$ respectively. Here J denote matrix with all entries equal to one and I denote the unit matrix.

Lemma 2.4.[4] Let A = $\begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ be a 2×2 block symmetric matrix. Then the eigenvalues of A are that of $M_1 + M_2$ together with $M_1 - M_2$.

Theorem 2.5. [6] Let D be the distance matrix of a connected regular graph G on 'n' vertices with its distinct eigenvalues $k = \eta_1, \eta_2, \ldots, \eta_n$. Then there exists a polynomial $P(x) = n\,\frac{(x-\eta_2)(x-\eta_3)\ldots(x-\eta_n)}{(k-\eta_2)(k-\eta_3)\ldots(k-\eta_n)}$ such that P(D) = J where J is the square matrix of order n whose all entries are one and k is the unique sum of each rows of D.

Product of graphs defined by *Yeh* and *Gutman* in [14] are as follows:

(1) The Cartesian product $G_1 \times G_2$:

$V(G_1 \times G_2) = V(G_1) \times V(G_2)$;

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \times G_2$ are adjacent if and only if either $[u_1 = v_1, (u_2, v_2) \in E(G_2)]$ or $[u_2 = v_2, (u_1, v_1) \in E(G_1)]$.

(2) The composition (lexicographic product) $G_1[G_2]$:

$V(G_1[G_2]) = V(G_1) \times V(G_2)$;

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1[G_2]$ are adjacent if and only if either $[u_1 = v_1, (u_2, v_2) \in E(G_2)]$ or $[(u_1, v_1) \in E(G_1)]$.

(3) The Kronecker (tensor) product $G_1 \otimes G_2$:

$V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$;

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \otimes G_2$ are adjacent if and only if $[(u_1, v_1) \in E(G_1)]$ and $[(u_2, v_2) \in E(G_2)]$.

(4) The join $G_1 \nabla G_2$:

$V(G_1 \nabla G_2) = V(G_1) \cup V(G_2)$;

$E(G_1 \nabla G_2) = E(G_1) \cup E(G_2) \cup \{(u_1, u_2) \mid u_1 \in V(G_1), u_2 \in V(G_2)\}$

3. *VDC*- spectra of some graphs

In this section we find the *VDC* -spectrum of join of two graphs, cartesian product of an arbitrary graph with K_2, double graph, lexicographic product of a graph with K_2, double odd graph and extended double cover graph.

3.1. Join of two graphs $G_1 \nabla G_2$

Theorem 3.1. For $i = 1, 2$ let G_i be a r_i regular graph on n_i vertices with adjacency spectrum $\lambda_1(G_i) \geq \lambda_2(G_i) \geq \ldots \geq \lambda_n(G_i)$ and diameter 2 then the *VDC* - spectrum of $G_1 \nabla G_2$ consists of $\lambda_i(G_1) - n + 2$ for $i = 2, 3 \ldots, n_1$,

$\lambda_j(G_2) - n + 2$ for $j = 2, 3, \ldots, n_2$ and two roots of the equation;

$x^2 - ((n-2)^2 + r_1 + r_2)x + ((n-2)(n_1 - 1) + r_1)((n-2)(n_2 - 1) + r_2)$

$- n_1 n_2 (n-1)^2 = 0.$ where $n = n_1 + n_2$.

Proof: From the definition of join of two graphs, the distance matrix of the join $G_1 \nabla G_2$ is

$$D(G_1 \nabla G_2) = \begin{bmatrix} D_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & D_2 \end{bmatrix},$$

where D_1 and D_2 are the distance matrices of G_1 and G_2 respectively and J is the matrix of all entries equal to 1.

The VDC - matrix of $G_1 \nabla G_2$,

$$B = \begin{bmatrix} n(J-I)_{n_1} & n\,J_{n_1 \times n_2} \\ n\,J_{n_2 \times n_1} & n(J-I)_{n_2} \end{bmatrix} - \begin{bmatrix} D_1 & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & D_2 \end{bmatrix},$$

where $n = n_1 + n_2$.

Since $diam(G) = 2$, the distance matrix $D = A(G) + 2\,\overline{A(G)} = 2(J - I) - A(G)$

$$B = \begin{bmatrix} (n-2)(J-I)_{n_1} + A(G_1) & (n-1)\,J_{n_1 \times n_2} \\ (n-1)J_{n_2 \times n_1} & (n-2)(J-I)_{n_2} + A(G_2) \end{bmatrix}.$$

Since G_1 is r_1 - regular, it has an eigenvector $\mathbf{1}_{n_1}$, a vector with all entries equal to 1, corresponding to the eigenvalue r_1. All other eigenvectors are orthogonal to $\mathbf{1}_{n_1}$. Let $\lambda(G_1)$ be an eigenvalue of the adjacency matrix $A(G_1)$ of G_1 with eigenvector X such that $\mathbf{1}^T X = 0$. Then $(X, 0)^T$ is an eigenvector corresponding to the eigenvalue $-(n-2) + \lambda(G_1)$. This is because

$$\begin{bmatrix} (n-2)(J-I)_{n_1} + A(G_1) & (n-1)\,J_{n_1 \times n_2} \\ (n-1)J_{n_2 \times n_1} & (n-2)(J-I)_{n_2} + A(G_2) \end{bmatrix} \begin{bmatrix} X \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left(-(n-2) + \lambda(G_1)\right)X \\ 0 \end{bmatrix}$$

$$= (\lambda(G_1) - n + 2) \begin{bmatrix} X \\ 0 \end{bmatrix}.$$

Similarly $(0, Y)^T$ is also an eigenvector corresponding to $\lambda(G_2)$. In an equivalent manner we can prove $-(n-2) + \lambda(G_2)$ is an eigenvalue of $A(G_2)$ and the corresponding eigenvectors are $(0, Y)^T$.

In this way we obtain eigenvectors of the form $(X, 0)^T$ and $(0, Y)^T$ all orthogonal to $(\mathbf{1}_{n1}, 0)^T$ and $(0, \mathbf{1}_{n2})^T$. Thus we obtain $n_1 - 1 + n_2 - 1 = n_1 + n_2 - 2$ eigenvalues of B.

The remaining two vectors of B are of the form $(\alpha\mathbf{1}, \beta\mathbf{1})$ for $(\alpha, \beta) \neq 0$.

Let x be the eigenvalue of B with eigenvector τ.

Then from $B\tau = x\tau$ we get

$$(n-2)(n_1 - 1)\alpha + r_1\alpha + (n-1)n_2\beta = x\alpha \quad (1)$$

$$n_1(n-1)\alpha + (n-2)(n_2 - 1)\beta + r_2\beta = x\beta \quad (2)$$

By solving equations (1) and (2) we get the remaining two eigenvalues.

3.2. Cartesian product $G \times K_2$

Theorem 3.2. Let G be a distance regular graph with distance spectrum $\{\eta_1, \eta_2, \ldots, \eta_n\}$ and distance regularity k. Then the *VDC* - spectrum of $G \times K_2$ is

$$VDCSpec(G \times K_2) = \begin{pmatrix} 4n^2 - 3n - 2k & -2(n+\eta_i) & -n & -2n \\ 1 & 1 & 1 & n-1 \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

Proof: From the definition of Cartesian product, we have distance matrix of $G \times K_2$ is

$$D(G \times K_2) = \begin{bmatrix} D & D+J \\ D+J & D \end{bmatrix}.$$

VDC matrix of $G \times K_2$ is

$$VDC(G \times K_2) = \begin{bmatrix} 2n(J-I) & 2nJ \\ 2nJ & 2n(J-I) \end{bmatrix} - \begin{bmatrix} D & D+J \\ D+J & D \end{bmatrix}$$
$$= \begin{bmatrix} 2n(J-I) - D & (2n-1)J - D \\ (2n-1)J - D & 2n(J-I) - D \end{bmatrix}.$$

By Lemma 2.4 the eigenvalues of $G \times K_2$ are those of $2n(J - I) - D + (2n - 1)J - D$ and $2n(J - I) - D - (2n - 1)J + D$.

i.e. the eigenvalues of $(4n - 1)J - 2nI - 2D$ and $J - 2nI$.

Using Theorem 2.5 we get the required spectrum.

Corollary 3.3. If G is a r - regular graph with diameter 2 and adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the *VDC* - spectrum of $G \times K_2$ is

$VDCSpec(G \times K_2)$

$$= \begin{pmatrix} 4n^2 - 7n - 2r + 4 & 2(\lambda_i + 2 - n) & -n & -2n \\ 1 & 1 & 1 & n-1 \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

3.3. Double Graph $D_2(G)$

Definition 3.1. [7] Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Take another copy of G with the vertices denoted by $\{u_1, u_2, \ldots, u_n\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$, the neighborhood of v_i, in G for each i. The resulting graph is called the double graph of G and is denoted by $D_2(G)$.

Theorem 3.4. Let G be a distance regular graph with distance spectrum $\{\eta_1, \eta_2, \ldots, \eta_n\}$ and distance regularity k. Then the *VDC* - spectrum of $D_2(G)$ is

$$VDCSpec(D_2(G)) = \begin{pmatrix} 4n^2 - 2n - 2 - 2k & -2(n+1+\eta_i) & -2(n-1) \\ 1 & 1 & n \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

Proof: From the Definition 3.1, the distance matrix of $D_2(G)$ is

$$D(D_2(G)) = \begin{bmatrix} D & D+2I \\ D+2I & D \end{bmatrix}.$$

VDC matrix of $D_2(G)$ is

$$VDC(D_2(G)) = \begin{bmatrix} 2n(J-I) - D & 2nJ - D - 2I \\ 2nJ - D - 2I & 2n(J-I) - D \end{bmatrix}.$$

By Lemma 2.4 the eigenvalues of $D_2(G)$ are those of $2n\,(J - I) - D + 2nJ - D - 2I$ together with $2n\,(J - I) - D - 2nJ + D + 2I$.
ie the eigenvalues of $4nJ - 2(n+1)I - 2D$ and $-2(n-1)I$.
Hence the theorem follows using Theorem 2.5.

Corollary 3.5. If G is an r - regular graph with diameter 2 and adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the VDC - spectrum of $D_2(G)$ is

$$VDCSpec(D_2(G)) = \begin{pmatrix} 4n^2 - 6n + 2r + 2 & -2(\lambda_i - n + 2) & -2(n-1) \\ 1 & 1 & n \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

3.4. Lexicographic product of G with K_2

Theorem 3.6. Let G be a distance regular graph with distance spectrum $\{\eta_1, \eta_2, \ldots, \eta_n\}$ and distance regularity k. Then the VDC - spectrum of the lexicographic product of G with K_2, $G[K_2]$, is

$$VDCSpec(G[K_2]) = \begin{pmatrix} 4n^2 - 2n - 1 - 2k & -(2n + 1 + 2\eta_i) & -(2n-1) \\ 1 & 1 & n \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

Proof: From the definition of lexicographic product, we have the distance matrix of $G[K_2]$ is

$$D(G[K_2]) = \begin{bmatrix} D & D + I \\ D + I & D \end{bmatrix}.$$

VDC matrix of $G[K_2]$ is

$$VDC(G[K_2]) = \begin{bmatrix} 2n(J - I) - D & 2nJ - D - I \\ 2nJ - D - I & 2n(J - I) - D \end{bmatrix}$$

By Lemma 2.4 the eigenvalues of $G[K_2]$ are those of $4nJ - (2n+1)I - 2D$ and $-(2n-1)I$. Remaining proof follows from Theorem 2.5.

Corollary 3.7. If G is a r - regular graph with diameter 2 and adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the VDC - spectrum of $G[K_2]$ is

$$VDCSpec(G[K_2]) = \begin{pmatrix} 4n^2 - 6n + 2r + 3 & 2\lambda_i - 2n + 3 & -(2n-1) \\ 1 & 1 & n \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

3.5. Double odd graph, DO(r)

Let n and r be two fixed integers. Consider the collection of integers S = {1, 2, 3, …, n} and $\binom{n}{r}$ denote the number of r – subsets of S. The graph $J(n; r; i)$ with fixed integers n, r and i is defined on the vertex set $\binom{n}{r}$ such that two vertices T_1 and T_2 are adjacent iff $|T_1 \cap T_2| = r - i$.

For $i = 1$ the graph $J(n; r; 1) = J(n, r)$ is called the Johnson graph. The Kneser graph K (n, r) is the Johnson graph $J(n; r; r)$ and the odd graph O(r) = K (2r +1, r). A Double odd graph DO(r) is a graph whose vertices are r - element or (r + 1) - element subset of $\{1, 2, \ldots, 2r + 1\}$. Two vertices T_1 and T_2 are adjacent iff $T_1 \subset T_2$ or

$T_2 \subset T_1$. Also, the Double odd graph can be constructed as the Kronecker product of O(r) with the path P_2.
For more details see [1].

Theorem 3.8. [3] The distance spectrum of Johnson graph *J(n,r)* is given by

$$Spec_D(J(n,r)) = \begin{pmatrix} s(n,r) & 0 & \frac{-s(n,r)}{n-1} \\ 1 & \binom{n}{r}-n & n-1 \end{pmatrix},$$

where, $s(n,r) = \sum_{j=0}^{r} j \binom{r}{j}\binom{n-r}{j}$.

Theorem 3.9. [1] Let $J(2r+1,r)$ be the Johnson graph of order $N = \binom{2r+1}{r}$. Let $r \geq 2$, the distance spectrum of the Double odd graph DO(r) is
Spec$Spec_D(DO(r)) =$

$$\begin{pmatrix} (2r+1)N & 0 & \frac{-s(2r+1,r)}{n-1} & -(2r+1)N+4s(2r+1,r) \\ 1 & 2N-2r-2 & 2r & 1 \end{pmatrix},$$

where, $s(n,r) = \sum_{j=0}^{r} j \binom{r}{j}\binom{n-r}{j}$.

Theorem 3.10. Let G be an arbitrary graph with distance spectrum $\{\eta_1, \eta_2, \ldots, \eta_n\}$. Then the *VDC* - spectrum of the double odd graph, DO(r), is
VDCSpec(DO(r)) =

$$\begin{pmatrix} N(4N-2r-3) & -2N & (2r-1)N-4s & -(2N-\frac{2s}{r}) \\ 1 & 2(N-r-1) & 1 & 2r \end{pmatrix},$$

where, $N = \binom{2r+1}{r}$ and $s = s(2r+1,r) = \sum_{j=0}^{r} j \binom{r}{j}\binom{r+1}{j}$.

Proof: We have from Theorem 3.8 distance matrix of $DO(r)$ is

$$\begin{bmatrix} 2D & (2r+1)J-2D \\ (2r+1)J-2D & 2D \end{bmatrix},$$

where D is the distance matrix of the Johnson graph $J(2r+1,r)$ and J in the above matrix is a square matrix of order $2r+1$ having all entries equal to 1.
Hence the *VDC* matrix of DO(r) is

$$VDC(DO(r)) = \begin{bmatrix} 2N(J-I)-2D & (2N-2r-1)J+2D \\ (2N-2r-1)J+2D & 2N(J-I)-2D \end{bmatrix}.$$

By Lemma 2.4 the *VDC*-eigenvalues of $DO(r)$ are those of $(4N-2r-1)J-2N\boldsymbol{I}$ and $(2r+1)J-2N\boldsymbol{I}-4D$.
By Theorem 2.5 the eigenvalues corresponding to $(4N-2r-1)J-2N\boldsymbol{I}$ are

$$\begin{pmatrix} (4N-2r-3)N & -2N \\ 1 & N-1 \end{pmatrix}. \qquad (3)$$

Since D is the distance matrix of the Johnson graph $J(2r+1,r)$, from Theorem 3.8,

$$Spec_D(J) = \begin{pmatrix} s(2r+1,r) & 0 & \frac{-s(2r+1,r)}{2r} \\ 1 & N-(2r+1) & 2r \end{pmatrix}, \quad (4)$$

where, $N = \binom{2r+1}{r}$ and $s = s(2r+1,r) = \sum_{j=0}^{r} j \binom{r}{j}\binom{n-r}{j}$.

Using Theorem 3.9 and Equation (4), the eigenvalues corresponding to $(2r+1)J - 2NI - 4D$ is

$$\begin{pmatrix} (2r-1)N - 4s(2r+1,r) & -2N & -(2N - \frac{2s(2r+1,r)}{r}) \\ 1 & N-(2r+1) & 2r \end{pmatrix}. \quad (5)$$

Combining Equations (3) and (5) we get the required *VDC* – spectrum of DO(r).

3.6. Extended double cover graph

The extended double cover of a graph was first introduced in 1986 by *Alon* [2] in connection with the study of networks.

Definition 3.2. [2] Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The extended double cover of G, denoted by $D^*(G)$, is the bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i and y_i are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G.

Theorem 3.11. If G is a r - regular graph on n vertices with diameter 2 and adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. The *VDC* - spectrum of G^*, the extended double cover of G is

VDCSpec(G^*) =

$$\begin{pmatrix} 4n^2 - 7n + 2r + 4 & -(n+2r) & 2\lambda_i - 2n + 4 & -2(n+\lambda_i) \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

for $i = 2, 3, \ldots, n$.

Proof: Let A be the adjacency matrix of G. Since $diam(G) = 2$, we have $D(G) = A(G) + 2\,\overline{A(G)}$. Then by the definition of G^*, the distance matrix of G^* can be written as,

$$D(G^*) = \begin{bmatrix} 2(J-I) & 3J - 2I - 2A \\ 3J - 2I - 2A & 2(J-I) \end{bmatrix}.$$

VDC matrix of G^* is

$$VDC(G^*) = \begin{bmatrix} (2n-2)(J-I) & (2n-3)J + 2I + 2A \\ (2n-3)J + 2I + 2A & (2n-2)(J-I) \end{bmatrix}.$$

By Theorem 3.5,. the eigenvalues of G^* are those of $(2n-2)(J-I) + (2n-3)J + 2I + 2A = (4n-5)J + (4-2n)I + 2A$ and $(2n-2)(J-I) - (2n-3)J - 2I - 2A = J - 2nI - 2A$.

Using Theorem 2.5 we get the required result.

Theorem 3.12. Let G be an A - integral graph with diameter atmost 2, then the following class of graphs are *VDC* - integral.

(1) Cartesian product of G with K_2 : $G \times K_2$

(2) Double graph : $D_2(G)$

(3) Lexicographic product of G with K_2 : $G[K_2]$

(4) Extended double cover graph : G^*

Proof: From Corollaries 3.3, 3.5, 3.7 and Theorem 3.11, it is clear that if G is A-integral then the VDC - spectrum of $G\times K_2$, $D_2(G)$, $G[K_2]$ and G^* consists only of integers.

4. *VDC*–energy

Energy of the graph G is introduced by Gutman [9]. Research is going on this field and many results obtained in this regard. For more details see [9 - 13].

Definition 4.1. [12] VDC - energy is the sum of the absolute values of the eigenvalues of the vertex distance complement matrix. It is denoted by $VDC\,E(G)$. If $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are the VDC - spectrum of a graph G then the VDC - energy is,

$$VDCE(G) = \sum_{i=1}^{n} |\theta_i|.$$

Theorem 4.1. Let G be a connected r - regular graph on n vertices with a diameter two. Then VDC - energy of $G \times K_2$ is

(i) The second largest eigenvalue of G is less than $n-2$ then,

$$VDCE(G\times K_2) = 2(4n^2 - 7n + 2r + 4)$$

(ii) The smallest adjacency eigenvalue of G is greater than or equal to $n-2$ then,

$$VDCE(G\times K_2) = 2n(2n-1)$$

Proof: Since G is r - regular, by Theorem 2.5 $4n^2 - 7n - 2r + 4$, the largest VDC - eigenvalue of $G \times K_2$, and is positive.

(i) By assumption $\lambda_i + 2 - n < 0, \text{ for } i = 2, 3, \ldots, n$.

Hence from Corollary 3.3 it is clear that $4n^2 - 7n + 2r + 4$ is the only positive VDC - eigenvalue of $G \times K_2$. Since the diagonal entries of $VDC(G)$ are zero, algebraic sum of the VDC - eigenvalues of any graph is zero. Therefore,

$$\begin{aligned} VDCE(G\times K_2) &= 2 \times (4n^2 - 7n + 2r + 4) \\ &= 2(4n^2 - 7n + 2r + 4). \end{aligned}$$

(ii) If $\lambda_i + 2 - n \geq 0$ then from Corollary 3.3, $-n$ and $-2n$ are the only negative eigenvalue of $VDC(G \times K_2)$ which repeats 1 and $n-1$ times respectively.

$$\begin{aligned} VDCE(G \times K_2) &= 2[n \times 1 + 2n \times (n-1)] \\ &= 2n(2n-1). \end{aligned}$$

Theorem 4.2. Let G be a connected r - regular graph on $n > 2$ vertices with diameter 2. Then VDC - energy of the double graph $D_2(G)$ is

(i) The second largest eigenvalue of G is less than $n-2$ then,

$$VDCE(D_2(G)) = 2(4n^2 - 6n + 2r + 2).$$

(ii)The smallest adjacency eigenvalue of G is greater than or equal to $n-2$ then,

$$VDCE(D_2(G)) = 2n(2n-1).$$

Proof: We have $(4n^2 - 6n + 2r + 2)$ is the largest VDC - eigenvalue of $D_2(G)$. By Theorem 2.5, it is positive.

(i) By assumption $\lambda_i - n + 2 < 0$ for all $i = 2, 3, \dots, n$.

Hence from Corollary 3.5 it is clear that $-2(n - 1)$ is the only negative VDC - eigenvalues of $D_2(G)$ which repeats n times. Therefore,

$$VDCE(D_2(G)) = 2 \times [n \times 2(n - 1)]$$
$$= 4n(n - 1).$$

(ii) If $\lambda_i - n + 2 \geq 0$, from Corollary 3.5, $4n^2 - 6n + 2r + 2$ is the only positive VDC - eigenvalue of $D_2(G)$. Therefore,

$$VDCE(D_2(G)) = 2 \times (4n^2 - 6n + 2r + 2)$$
$$= 4(2n^2 - 3n + r + 1).$$

Theorem 4.3. Let G be a connected r - regular graph on n vertices with diameter 2. Then VDC - energy of the lexicographic product of G with K_2, $G[K_2]$, is

(i) The second largest eigenvalue of G is less than $\frac{2n-3}{2}$ then,

$$VDCE(G[K_2]) = 2[4n^2 - 6n + 2r + 3].$$

(ii) The smallest adjacency eigenvalue of G is greater than or equal to $\frac{2n-3}{2}$ then,

$$VDCE(G[K_2]) = 2n(2n - 1).$$

Proof: The proof of the Theorem is similar in lines that of the above Theorem.

Theorem 4.4. Let G be a connected r - regular graph with $n > 4$ vertices. Then VDC - energy of G^*, the extended double cover of G, is

(i) The second largest eigenvalue of G is less than $n - 4$ then,

$$VDCE(G^*) = 2[4n^2 - 7n + 2r + 4].$$

(ii)The smallest adjacency eigenvalue of G is greater than or equal to $n - 4$ then,

$$VDCE(G^*) = 2n(2n - 1).$$

Proof: Since G is r - regular, $4n^2 - 7n - 2r + 4$ is positive. Also $-r \leq \lambda_i \leq r$ for $i = 2, 3, \dots, n$.

So $-n < -r \leq \lambda_i \Rightarrow n + \lambda_i > 0$ for $i = 2, 3, \dots, n$.

(i) By assumption $\lambda_i + 2 - n < 0$, for $i = 2, 3, \dots, n$.

Hence from Theorem 3.11, it is clear that $4n^2 - 7n - 2r + 4$ is the only positive VDC – eigenvalue of G^*. Since the diagonal entries of $VDC(G)$ are zero, algebraic sum of the VDC – eigenvalues of any graph is zero. Therefore,

$$VDCE(G^*) = 2 \times (4n^2 - 7n - 2r + 4)$$
$$= 2(4n^2 - 7n + 2r + 4).$$

(ii) If $\lambda_i + 2 - n \geq 0$ then from Theorem 3.11 $-(n + 2r)$ and $-2(\lambda_i + n)$, $i = 2, 3, \dots, n$ are the only negative VDC - eigenvalue of G^*.

We have $\sum_{i=1}^{n} \lambda_i = 0 \Rightarrow \sum_{i=2}^{n} \lambda_i = -r$,

$$VDCE(G^*) = 2[(n + 2r) \times 1 + 2\sum_{i=2}^{n}(n + \lambda_i)$$
$$= 2[n + 2r + 2((n - 1)n - r)]$$
$$= 2n(2n - 1).$$

Definition 4.2. Two connected graphs G_1 and G_2 are said to be vertex distance complement equienergetic or VDC - equienergetic if $VDCE(G_1) = VDCE(G_2)$.

Theorem 4.5. Let G be a r - regular graph with diameter 2, then $G \times K_2$ and G^* are VDC – equienergetic graphs.

Proof: Proof of the theorem follows from Theorems 4.1 and 4.2.

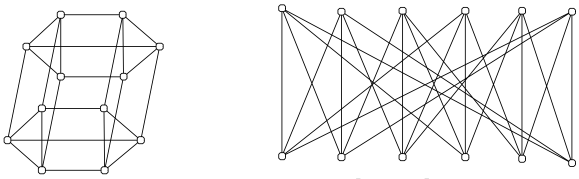

Figure 1: *VDC*-equienergetic graphs with *VDC* - energy 224.

5. Conclusion

The spectral graph theory has various applications in the field of science like theoretical chemistry, quantum mechanics, statistical physics, computer, information science etc. There are mainly two models QSPR and QSAR which are used for the study of molecular design of chemical compound. The vertex distance complement matrix is one of the important sources of structural description for QSPR and QSAR models. In this paper we construct the *VDC*- spectrum and *VDC* - energy of some class of graphs. Here we discuss some infinite family of *VDC* - integral graphs. As an application we can give the *VDC* -energy of $G \times K_2$, $G[K_2]$, $D_2(G)$ and the extended double cover graph. Also we proved that the cartesian product $G \times K_2$ and the extended double cover graph of G are *VDC*-equienergetic graphs.

Acknowledgements. The authors are thankful to the reviewer for their comments and suggestions for improving the quality of this paper. Also the author thankful to the University Grants Commission of Government of India for providing fellowship under the FDP in the XII^{th} plan.

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