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Numerical Study of Fuzzified Boundary Value Problem for Couette type Flow of Fluid Mechanics

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Abstract. In this paper**,** an attempt has been made to fuzzify some parameters and variables appeared in equations of fluid mechanics and then defined the fuzzy derivatives using Zadeh extension principle. The differential equations governing the Couetee type flow which is one of the fundamental boundary value problems is also fuzzified. The fuzzified boundary value problem has been discritized using Crank Nicolson scheme. Finally, numerical solution is carried out by developing computer codes for the problem. The crisp solution and mid value solution of the triangular fuzzified system of equations are in good agreement.

Keywords: Triangular fuzzy number, Reynolds number, couette flow

AMS Mathematics Subject Classification (2010): 76Mxx

1. Introduction

In day to day life we encounter many situation most of them are fuzzy in nature [1,2,3,4]. To tackle such fuzziness in many application related problems mathematical models or mathematical equation are derived. Many such problems are models to differential equation having some prescribe boundary condition. When a real world problem is transformed value problem of ordinary differential equations, or a system of differential equations, we cannot be usually sure that the model is perfect to illustrate the system. For example, the initial value problem (IVP) $\frac{dy}{dx} = f(x, y),\ x(y_0) = y_0$ the initial value may not be known exactly and the function *f* may contain uncertain parameter. If they are estimated through certain measurements, they are necessarily subject to errors. The analysis of the effect of these errors leads to the study of the qualitative behavior of the solutions of the IVP. If the nature of the error is random then we can discuss, random differential equation with random initial data. However, if the underlying structure is not probabilistic due to subjective choices, it would be natural to employ fuzzy differential equations. The solution and behaviors of such differential equation are fuzzy in nature. So it is important to fuzzifed the differential equations appeared in many branches of science and technology such as Physics, Fluid Mechanics, Mechanical Engineering and

Civil Engineering. Such models are successively applied in the fields of civil engineering [5], population model [6] and as well in modeling of hydraulic [7].

The first fuzzy valued function was developed by Chang and Zadeh [8]. After that Dubois and Prade [9], Puri and Ralescu [10], Kaleva [11] Seikkala [12] etc. came with their own approaches. Subsequently lots of Researcher has done the work on the field of fuzzy differential equation see e.g.[12,13, 14].

For the solution of such differential equation many researchers investigated numerical solution with the numerical methods such as Runge-Kutta [15], Fuzzy transform [16], difference methods [17], shooting Method [18] and many more.

Since couette type flow problems are a fundamental flow problem in Fluid Dynamics having many applications drawn the attention of many works in [17,18,19]. In this paper an attempt has been made to fuzzify the differential equation that represents the couette type flow and to solve the fuzzified boundary value problems. The fuzzify the differential equation after discritisation using Crank-Nickolson methods has been solved by developing computer codes for the fuzzified boundary value problem. It is interesting to note that the mid solution is almost identical with that the crisp solution.

2. Preliminaries

(i) Basic of fuzzy set theory

Definition 2.1. ***Fuzzy set*** : Let X is a collection of objects denoted generally by x, then a fuzzy set of ordered pairs Ã in X is a set of order pairs

$$\tilde{A}= \{ (x,\mu_{\tilde{A}}(x)) : x \in X\}$$

$\mu_{\tilde{A}}$ is called the membership function or grade of membership of x in $\tilde{A}$. The range of the membership function is a subset of the non-negative real number whose supremum is finite.

Definition 2.2. ***Height of a fuzzy set:*** It is defined as the largest membership grade obtained by any element of a fuzzy set. i.e.

$$h(A) = sup_{x\in X}\mu_{\mathrm{A}}(x).$$

Definition 2.3. ***Normal fuzzy set:*** A fuzzy set A is said to be normal if $h(A) = 1$.

Definition 2.4. ***Triangular fuzzy set:*** A fuzzy set is called triangular if the membership function of the set is given by

$$\mu_{\mathrm{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \end{cases}$$

Simply it can be express as $\widetilde{A} = [a, b, c]$

Definition 2.5. ***Operation of triangular fuzzy set:***

$\tilde{A}_1 = [a_1, b_{1,} c_1]$ and $\tilde{A}_1 = [a_2, b_{2,} c_2]$ are two triangular fuzzy number then Operation of these two are defined as

(a) Addition: $\tilde{A}_1 + \tilde{A}_2 = [a_1, b_{1,} c_1] + [a_2, b_{2,} c_2] = [a_1 + a_2, b_1 + b_{2,} c_1 + c_2]$

(b) Subtraction: $\tilde{A}_1 - \tilde{A}_2 = [a_1, b_{1,} c_1] - [a_2, b_{2,} c_2] = [a_1 - a_2, b_1 - b_{2,} c_1 - c_2]$

(c) Scalar Multiplication: $\sigma\tilde{A}_1 = [\sigma a_1, \sigma b_{1,}\sigma c_1]$

(d) Division: $\frac{\tilde{A}_1}{\tilde{A}_2} = \left[\min T, \frac{b_1}{b_2}, max\ T\right]$, where $T = \{\frac{a_1}{a_2}, \frac{c_1}{a_2}, \frac{c_1}{c_2}, \frac{c_1}{c_2}\}$

(e) Multiplication: $\tilde{A}_1 * \tilde{A}_2 = [mina_i b_{i\,,} product\ of\ mid\ point, \max a_i b_i]$

Definition 2.6. *Zadeh extension principle:*
When a crisp function $f: X \to Y$is said to be fuzzified when it is extended to act on fuzzy set defined on X and Y. i.e

$$\tilde{f}: \tilde{X} \to \tilde{Y}$$

And its inverse has the form $\tilde{f}^{-1}: \tilde{Y} \to \tilde{X}$

The extension principle state that for a given crisp function $f: X \to Y$ induces two functions $\tilde{f}$ and $\tilde{f}^{-1}$ which are defined above for which membership function are given by

$$[f(\tilde{A})](y) = sup_{x:y=f(x)}\mu_{\hat{A}}(x)$$

For all A ∈ $\tilde{X}$. And

$$[\tilde{f}^{-1}(B)](x) = \mu_B f(x)$$

For all B ∈ $\tilde{Y}$

(ii) Basic of dynamics

Definition 2.7. *Incompressible fluid:* A fluid is said to be incompressible if it requires a huge variation in pressure to produce some appreciable variation of its density. A fluid which is not incompressible is called compressible fluid.

Definition 2.8. *Laminar flow:* The flow in which each fluid particle followed a definite curve and the curve trace out any two distinct fluid particle can't intersect, is called Laminar flow.

Definition 2.9. *Steady and Unsteady flow:* A flow in which property and condition associated with the motion of the fluid are independent of time so that the flow pattern remain unchanged with the time is called steady flow, otherwise it is unsteady.

Mathematically for steady flow

$$\frac{\partial}{\partial t}\boldsymbol{P} = \boldsymbol{0},$$

For unsteady flow $\frac{\partial}{\partial t}\boldsymbol{P} \neq \boldsymbol{0}$,

where P may be any of velocity, density, pressure, temperature etc.

Definition 2.10. *Rotational and irrotational flows*: A flow in which fluid particle go on rotating about their own axes, while flowing, is said to be rotational. And the flow which is not rotational is called irrotational flow.

Definition 2.11. *Equation of continuity:* This is also known as equation of conservation of mass. In vector notation the equation is given by: $\frac{\partial p}{\partial t} + \nabla.(\rho\overrightarrow{q}) = 0.$

In Cartesian co-ordinate system the above equation is reduce to

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

Where $\vec{q}=u\hat{\imath}+v\hat{\jmath}+w\hat{k}$ and u, v, w are the velocity component of $\vec{q}$ along the $x, y\ and\ z$ direction.

Definition 2.12. ***Navier -Stokes equation:*** Navier stockes is a set of equation of motion for a viscous incompressible fluid. In Cartesian coordinates these are is given by

$$\rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right\}=\rho B_x-\frac{\partial p}{\partial x}+\mu\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\}$$

$$\rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right\}=\rho B_y-\frac{\partial p}{\partial y}+\mu\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\}$$

$$\rho\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\}=\rho B_z-\frac{\partial p}{\partial z}+\mu\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\}$$

Definition 2.13. ***Reynolds number:*** This is a dimensionless number which is defined as the ratio of inertia force by viscous force is known as Reynolds number.

Mathematically the expression for Reynolds number is

$$Re.=\frac{VL}{\vartheta}$$

where V and L are the characteristic velocity and length and $\vartheta=\frac{\mu}{\rho}$. Here the symbol indicates the usual meaning.

Couette flow: The couette flow is the flow problem of a viscous incompressible fluid between two parallel plates, one plate is at rest and the other is moving with a constant velocity. This is one of the problems for which exact solution of Navier-Stoke equation for one dimensional flow with given boundary conditions.

Objective of the problem: The main objective of these type problems is to obtain velocity, temperature, coefficient of skin friction and to estimate the rate of Heat Transfer .

Physical model of the problems: Consider a laminar flow Newtonian incompressible fluid between two infinite parallel plate which are kept at a distance h apart. We assuming that upper plate is moving in the direction of x- axis with constant velocity u_o and $y-$ axis perpendicular to x and fluid flow properties are independent of z. Also we have taken that there is no external force.

u=u_0

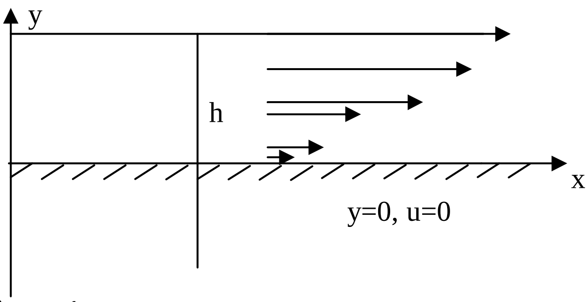


Figure 1:

The flow is considered to be unsteady, laminar and single phase. The governing equation and the boundary condition of this flow is given by the equation.

Mathematical model: The governing Equation and boundary condition of the above problem is given by the following

Equation of continuity

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \tag{1}$$

Navier Stockes equation

$$\rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right\}=\rho B_x-\frac{\partial p}{\partial x}+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right\} \tag{2}$$

$$\rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right\}=\rho B_y-\frac{\partial p}{\partial y}+\mu\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\} \tag{3}$$

$$\rho\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\}=\rho B_z-\frac{\partial p}{\partial z}+\mu\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\} \tag{4}$$

Assumption under coquette flow are given by

(a) The plates are of infinite extent in x-direction, so all quantities are independent of x. i.e. $\boldsymbol{\frac{\partial}{\partial x}(\ T)=0}$. Where T may be velocity, Temperature etc.

(b) Similarly as there is no body force involved in the motion so the physical quantity all are equivalent to zero, i.e. we have $\boldsymbol{B_x=B_y=B_z=0}$

(c) There is no motion along $z-$ axis so we have $\boldsymbol{\frac{\partial}{\partial z}(T\ )=0\ , w=0.}$ Where T may be velocity, Temperature etc

Under these assumption equations (1),(2),(3) and (4) become

$$\boldsymbol{\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}} \tag{5}$$

The boundary condition: u(t,0)=0 and u(t,h)= u_o

To make the equation and the boundary conditions dimensionless let us assume

$u'=\frac{u}{U_0}\quad t'=\frac{t}{T_0}\qquad y'=\frac{y}{L_0}$

Then the above equation reduces to

$$\frac{\mu}{\rho L_0 U_0}\frac{\partial^2 u'}{\partial y'^2}=\frac{\partial u'}{\partial t'}$$

$$\Rightarrow\frac{\partial u'}{\partial t'}=\frac{1}{Re.}\frac{\partial^2 u'}{\partial y'^2} \tag{6}$$

where $Re.$ indicate the Reynold number.

The boundary conditions become: u(t',0)=0,
u(t',1)=1. (7)

Fuzzified the above equation and boundary condition we have,

$$\frac{\partial\hat{u}'}{\partial\hat{t}'}=\frac{1}{\widehat{Re}.}\frac{\partial^2\hat{u}'}{\partial\hat{y}'^2} \tag{8}$$

And the boundary condition $\hat{u}(t',0)=0$,
$\hat{u}(t',1)=1$ (9)

(For our simplicity we just remove the bar given in each variable. Example

u' $became$ $u, t'became$ t $etc.$) Then The above equation discrtized using Crank Nicolson scheme and the final finite difference equations become:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \frac{1}{Re.} * \frac{1}{2} * \frac{u_{i+1,j+1} + u_{i,j+1} - 2u_{i+1,j} - 2u_{i,j} - u_{i+1.j-1} + u_{i,j-1}}{\Delta y^2}$$

$$\Rightarrow u_{i+1,j} = u_{i,j} + \frac{\Delta t}{Re.} * \frac{1}{2} * \frac{u_{i+1,j+1} + u_{i,j+1} - 2u_{i+1,j} - 2u_{i,j} - u_{i+1.j-1} + u_{i,j-1}}{\Delta y^2} \quad (10)$$

where $i = 1,2,3, \dots\dots\dots\dots M$, $j = 1,2,3, \dots\dots\dots N$. And the boundary condition

$$u(i,1) = 0,$$
$$u(i, N+1) = 1 \quad (11)$$

Zedeh Extension principle we shall convert the above equation into fuzzyfied form we have

$$\Rightarrow U_{i+1,j} = U_{i,j} + \frac{\Delta T}{RE.} * \frac{1}{2} * \frac{U_{i+1,j+1} + U_{i,j+1} - 2U_{i+1,j} - 2U_{i,j} - U_{i+1.j-1} + U_{i,j-1}}{\Delta Y^2} \quad (12)$$

With the boundary condition $U(i,1) = 0,$
$$U(i, N+1) = 1 \quad (13)$$

Here all block letter are the fuzzified form of the respective variables.

3. Result and discussion

The equation for the crisp variable is given in the equation (6) with the boundary condition (7) and the fuzzified boundary value problem is given by (8) - (9).

The Crank-Nickolson Scheme for the discritized equation (6)-(7) is given in Equation (10)-(11). The discritized equations (8) – (9) after applying Zadeh extension principle are given by (12) and (13).

The discritized equations (10) - (11) and (12) - (13) are solved by an iterative scheme based on Gauss Saidel method for different time as well as Reynolds number. Solution is carried out by developing computer codes for the problem. The solution tables for Reynolds number Re= 10, t= 0 ; Re= 15, t= 0.2 ; Re= 20, t= 0.4; are given below.

Table I: Re. no.=10 and t=0.0

y	crispvalue u	midvalue U	leftvalue U	midvalue U	rightvalue U
0	0	0	0	0	0
0.1	0.000296	0.000295	-0.01376	0.000295	0.013876
0.2	0.000887	0.000887	-0.0206	0.000887	0.021591
0.3	0.002365	0.002365	-0.02199	0.002365	0.025805
0.4	0.006208	0.006208	-0.01926	0.006208	0.030703
0.5	0.01626	0.01626	-0.00983	0.01626	0.041145
0.6	0.042572	0.042572	0.015262	0.042572	0.067499
0.7	0.111456	0.111456	0.081489	0.111456	0.135929
0.8	0.291796	0.291796	0.258609	0.291796	0.314371
0.9	0.763932	0.763932	0.73317	0.763932	0.779979
1	1	1	1	1	1

Table II: Re. no.=10 and t=0.2

y	crispvalue u	midvalue U	leftvalue U	midvalue U	rightvalue U
0	0	0	0	0	0
0.1	0.000489	0.000489	-0.03092	0.000489	0.030831
0.2	0.001551	0.001551	-0.05108	0.001551	0.052328
0.3	0.004379	0.004379	-0.05903	0.004379	0.065548
0.4	0.011978	0.011978	-0.05606	0.011978	0.077694
0.5	0.03181	0.03181	-0.0382	0.03181	0.099387
0.6	0.081082	0.081082	0.009243	0.081082	0.149276
0.7	0.193754	0.193754	0.118299	0.193754	0.2611
0.8	0.411552	0.411552	0.336546	0.411552	0.472625
0.9	0.655631	0.655631	0.601393	0.655631	0.695913
1	1	1	1	1	1

Table III: Re. no.=15 and t=0.4

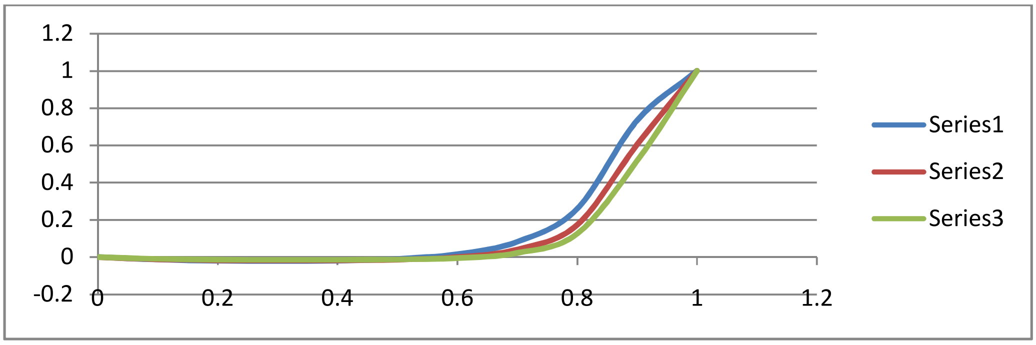
y	crispvalue u	midvalue U	leftvalue U	midvalue U	rightvalue U
0	0	0	0	0	0
0.1	0.000655	0.000655	-0.05705	0.000655	0.05643
0.2	0.002132	0.002132	-0.09804	0.002132	0.098917
0.3	0.006135	0.006135	-0.11831	0.006135	0.126385
0.4	0.016793	0.016793	-0.11911	0.016793	0.14827
0.5	0.043486	0.043486	-0.09714	0.043486	0.17965
0.6	0.104405	0.104405	-0.03824	0.104405	0.241746
0.7	0.224902	0.224902	0.081858	0.224902	0.359012
0.8	0.415971	0.415971	0.283714	0.415971	0.533649
0.9	0.686897	0.686897	0.597843	0.686897	0.760546
1	1	1	1	1	1

Table I, Table II and Table III are few example of the output of the programming developed for the said problems. Here we see that the mid value of the fuzzy solution and the crisp solution of the problems are in good agreement.

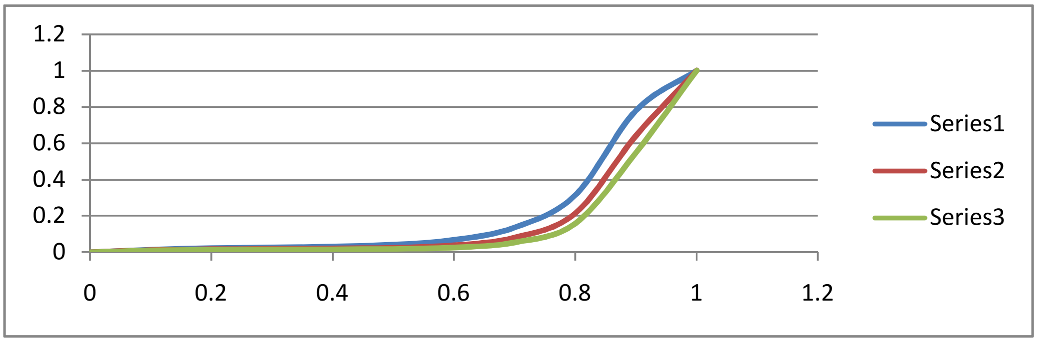
In the next part we draw some graph of the fuzzy solution which are obtain from the developed computer codes for the same problems. The graphs of fuzzy values of U against Y for different times and Renoylds number are presented in figures 2-9.

At t=0, Re.=10,15,20

(Left values) **Figure 2:**

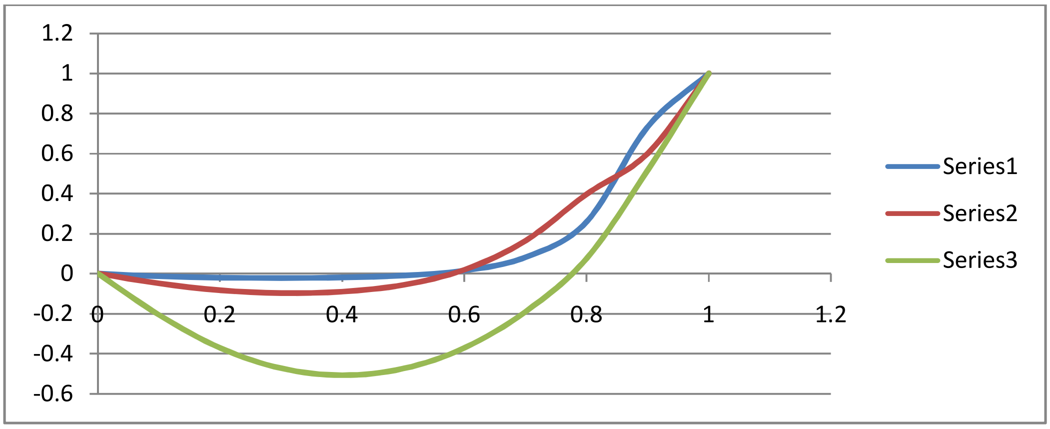


(Right values) **Figure 3:**



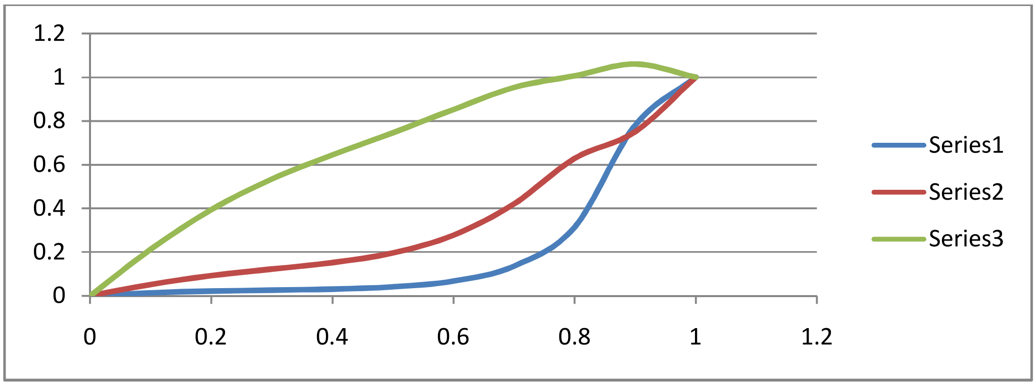
Re.10, t=0.00, 0.2,0.4

(left values) **Figure 4:**



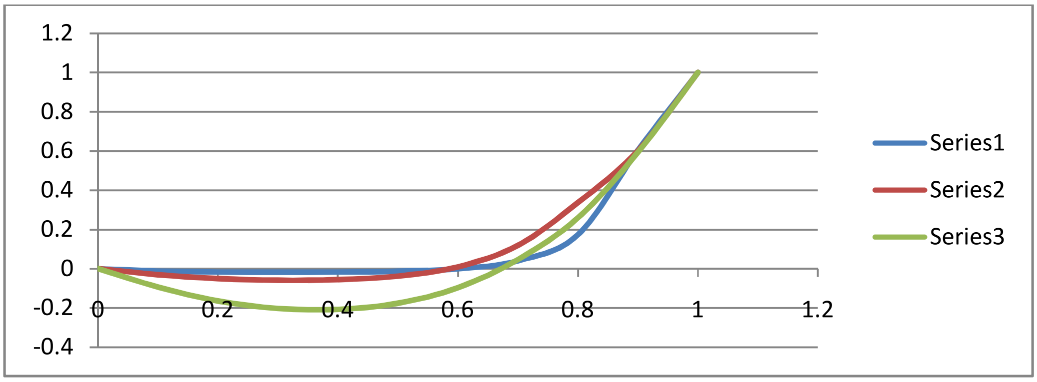
Re.10, t=0.00, 0.2,0.4

(right values) **Figure 5:**

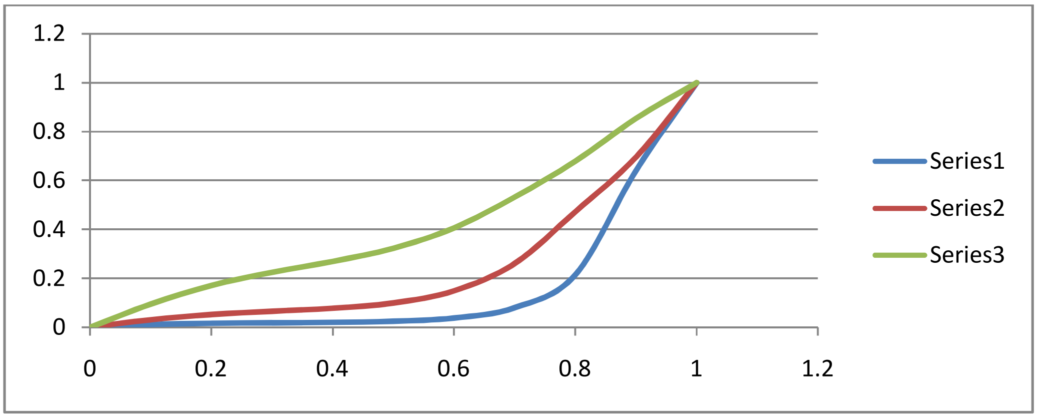


Re.15, t=0.00, 0.2,0.4

(left values) **Figure 6:**

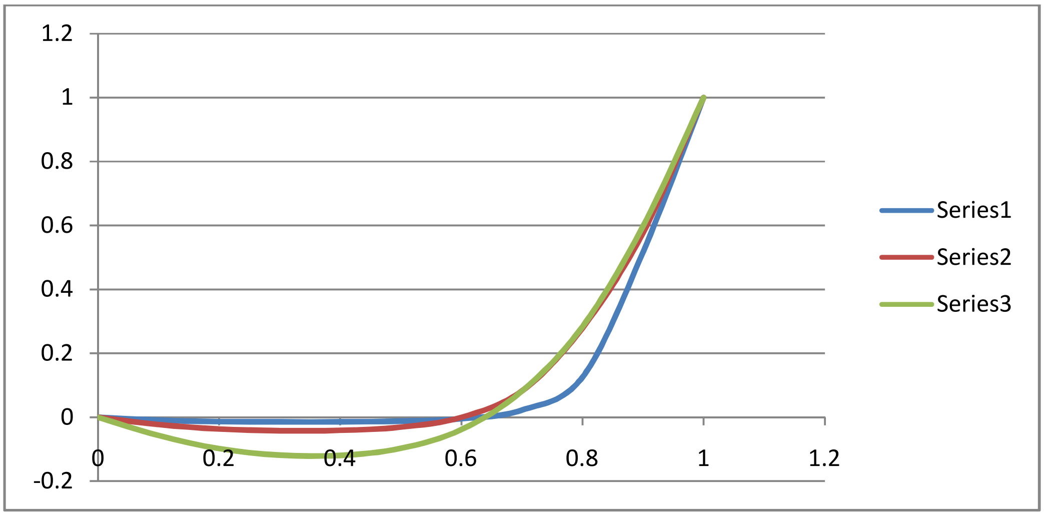


(Right values) **Figure 7:**

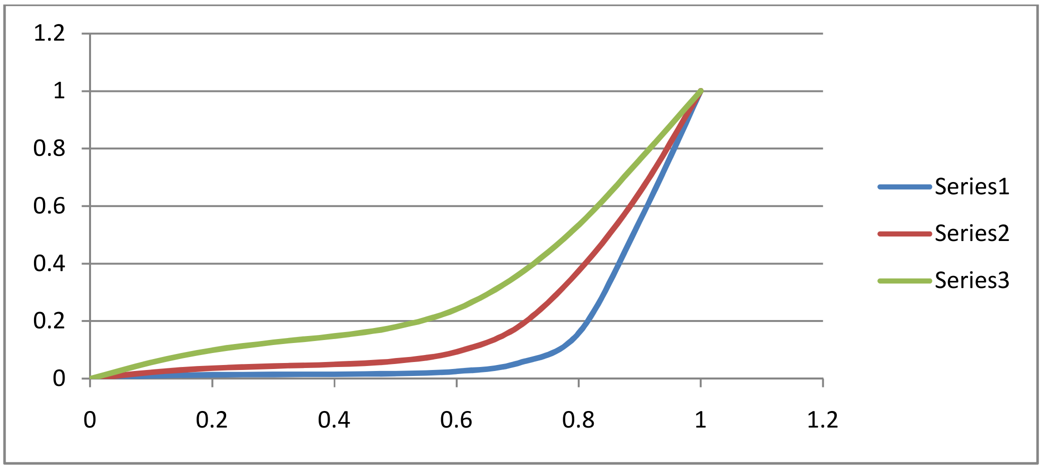


Re.20, t=0.00, 0.2, 0.4

(left values) **Figure 8:**



(Right values) **Figure 9:**



4. Conclusion

From the graphs we observed that when Reynolds Number increases the uncertainty decreases. From the physical model of Reynolds Number we know that increase of Reynolds Number indicate that the viscous force is less so flow pattern is uniform so clearly less chance of uncertainty will arise.

For all Reynolds Number within the interval of time [0,0.2] we get a point of inflection near y=0.6. When time $t \geq 0.4$ the uncertainty of the solution of the problem increases. For the given set of parameter within the interval [0,0.2] there is less uncertainty.

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REFERENCES

1. P.Murugadas and M.R.Thirumagal, Intuitionistic Q1-Fuzzy k-ideals of semi-ring, *Annals of Pure and Applied Mathematics*, 15(1) (2017) 89-95.
2. M.Pal, Fuzzy matrices with fuzzy rows and fuzzy columns, *Journal of Intelligent and Fuzzy Systems*, 30(1) (2015) 561-573
3. I.Silambarasan and S.Sriram, Bounded sum and bounded product of fuzzy matrices, *Annals of Pure and Applied Mathematics*, 14(3) (2017) 513-523.
4. S.Yahya Mohamed and N.Subashini, Connectivity in a bipolar fuzzy graph and its complement , *Annals of Pure and Applied Mathematics*, 14(3) (2017) 513-523.
5. M.Guo, X.Xue, and R.Li, Impulsive function differential inclusions and fuzzy population models, *Fuzzy Sets and System*, 138(3) (2003) 601-615.
6. M.Guo, X.Xue, and R.Li, The oscillation of delay differentials inclusions and fuzzy population model, *Mathematical and Computer Modeling*, 37(7-8) (2003) 651-658.
7. A.Bencsik, B.Bede, J.Tar and J.Fodor, Fuzzy differential equations in modeling hydraulic differentials servo cylinders, *Proccending of the 3rd Romanian-Hungarian joint Symposium on Applied Computational Intellidence (SACI'06*), Timisoara, Romania, 2006.
8. S.S.I.Chang and L.A.Zadeh, On fuzzy mapping and control, *'IEEE Transaction and System,Man and Cybernetics*, 2 (1982) 225-233.
9. D.Dubois and H.Prade, Towards fuzzy differential calculus. III. Differentiations, *Fuzzy Sets and System*, 8(3) (1982) 225-233.
10. M.L.Puri and D.A.Ralescu, Differentials of fuzzy function, *Journal of Mathematical Analysis and Application*, 91(2) (1983) 552-588.
11. O.Kaleva, Fuzzy differential equation, *Fuzzy Sets and System*, 24(3) (1987) 301-317.
12. S. Seikkala, On the fuzzy intial value problems, *Fuzzy Sets and System*, 35(3) (1990) 387-396.
13. O.Kaleva, The Cauchy problems for fuzzy differentials equations, *Fuzzy Sets and System*, 35(2) (1990) 389-396.
14. Y.Zhang, G.Wang and S.Liu, Frequently domain methods for solution of n-order fuzzy differentials equations, *Fuzzy Sets and System*, 94(2) (1998) 45-59.
15. F.Rabie, F.Ismail, A.Ahmadian and S.Salahshour, Numerical solution of fuzzy differentials equation using Improved Runge-Kutta Nystrom Method, *Research Article, Hindawi Publication Corporation, Mathematical Problems in Engineering*. Article ID 803462, 2013.
16. M. Afshar Kermani and F.Saburi, Numerical methods for fuzzy differential equations, *Applied Mathematical Sciences*, 1 (27) (2007) 1299-1309.
17. V.A.Romanov, Stability of plane-parallel Couette flow, *Funct. Anal. Appl.*, 7(2) (1973) 137-146.

18. R.D.Moser, A spectral numerical method for the Navier-Stokes equations with applications to Taylor-Couette flow, *Journal of Computation Physics*, 52(3) (1983) 524-544.
19. M.Avgousti and A.N.Beris, Non-axisymmetric modes in viscoelastic Taylor-Couette flow, *Journal of Non-Newtonian Fluid Mechanics*, 50(2–3) (1993) 225-251.