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On *p-h* Points and Completeness Property of a Partial Metric Space

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Abstract. Role of *d*-points in a metric space is well-known. The notion has been extended in a partial metric space (X, p) and the consequences of *p*-*h* points have been investigated with some applications in theory of fixed points.

Keywords: Partial metric spaces, completeness, *p*-*h* points, fixed point.

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1. Introduction

The notion of partial metric spaces, which allow non-zero self-distance, was introduced as a generalization of metric spaces by Matthews [14] in 1994 where he gave a generalization of Banach's contraction mapping theorem over a metric space. Since then, many researchers have worked on different aspects, in particular, in the realm of fixed point theory of partial metric spaces. For investigations in partial metric spaces, works of Valero and Oltra [9, 15], Romaguera [16], Altun et al. [6], Choudhury [2, 3] are noteworthy. In this connection it should also be mentioned that before the introduction of partial metric spaces, there were other generalizations of metric spaces, most notably 2-metric spaces and generalized metric spaces where fixed point theorems for contractive mappings had been investigated (see for example the work of Das et al. [4, 10] or Dey et al. [17]. Again in [1, 5, 11] one finds several fixed point theorems proved for operators involving Kannan contractions, weak contractions and ground space sometimes endowed with an associated graph. In particular, exclusively with Kannan contraction in fixed point theory, we have references like [8,12,13].

In the year 1977, Weston [7] defined *d*-point for a real-valued function$h$on a metric space (X, d) and obtained a characterization of completeness ofthe metric space (X, d). Further, he applied the result in fixed point theory over a complete metric space (X, d).

In any generalization of metric spaces, completeness has always been one of the most fundamental properties. In particular characterizations of completeness has been of much interest. In this paper, we have defined *p*-*h* point in the setting of partial metric spacesand studied the completeness of a partial metric space with the aid of such *p-h* points. Also, the concept of *p-h* points was utilized to deduce some fixed point theorems over complete metric spaces.

We recall a few definitions first.

A partial metric on $X$is a function $p: X \times X \to \mathbb{R}^+$(the set of nonnegative real numbers) satisfying conditions as under:
For all$x, y, z \in X$,

(i) $x = y$if and only if$p(x,x) = p(y,y) = p(x,y)$,
(ii) $p(x,x) \leq p(x,y)$,
(iii) $p(x,y) = p(y,x)$,
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

If p is a partial metric on the set X, (X,p) is known as a partial metricspace. Thus in a partial metric space (X,p), each point does not necessarily possess zero distance from itself. Of course, a metric space is a partial metric space while the converse is false.

Example 1.1. Take $\mathbb{N}$ = set of all natural numbers, and $p : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ is defined by

$$p(m,n) = \begin{cases} \frac{1}{n} \text{if } m = n, \\ \frac{1}{n} + \frac{1}{m} \text{if } m \neq n. \end{cases}$$

Then $(\mathbb{N}, p)$ is a partial metric space.

Now let us give a look into the topological aspects of a partial metric space (X,p).

If $x \in X$ and $\epsilon > 0$, then the set $B_\epsilon(x) = \{ y \in X : p(x,y) < p(x,x) + \epsilon \}$ is called a p-open ball in (X,p). By routine check up one finds $\{B_\epsilon(x)\}$,$x \in X$and $\epsilon > 0$ is a base to generate a topology τ_p called the partial metric topologyon X, and this topology τ_p is T_0.

We have the following definitions in a partial metric space (X,p).

Definition 1.2. A sequence $\{x_n\}$in a partial metric space (X,p) is said tobe a p-Cauchy sequence if $\lim_{m,n\to\infty} p(x_n, x_m)$exists.

Definition 1.3. A sequence $\{x_n\}$ in a partial metric space (X,p) is said tobe p-convergent at $x_0 \in X$ if $\lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$.

Definition 1.4. A partial metric space (X,p) is said to be complete if every p-Cauchy sequence in (X,p) p-converges to a point of X, i.e., if $\{x_n\}$ is p-Cauchy in (X,p), there is a point $x_0 \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0).$$

2. Main results

Before going into the main result, we give some basic definitions.

Definition 2.1. A function $h : (X, p) \rightarrow \mathbb{R}$ (with usual topology) is said to be p-lower semi continuous (p-1.s.c.) at $u \in X$ if given $\epsilon > 0$, there is a δ >0 such that

$$h(x) > h(u) - \epsilon \text{ for } x \in p\text{-}B_\delta(u),$$

or, equivalently, if $\{x_n\}$ is a sequence in (X, p) p-converging to u, then

$$\underline{\lim}_{n\to\infty} f(x_n) \geq f(u).$$

Also h is said to be a p-l.s.c. function on Xif it is so at every point of X.

Definition 2.2. Given a function $h : (X, p) \rightarrow \mathbb{R}$ (with usual topology) a point $x_0 \in X$ is said to be a p-h point if and only if for every $x \in X$with$x \neq x_0$,

$$h(x_0) - h(x) < p(x_0, x) - p(x, x)(> 0).$$

Theorem 2.3. Let (X, p) be a complete partial metric space and $h : (X, p) \rightarrow \mathbb{R}$ be a p-lower semi continuous function that is bounded below, then there is a p-h point in X.

Proof: Let $h : (X, p) \rightarrow \mathbb{R}$ be a p-lower semi continuous function that is bounded below where (X, p) is complete. We consider a member$x_1 \in X$; let x_1 be not a p-h point for $h$in X. Then we can find $x \in X$ with $x \neq x_1$ satisfying

$$h(x_1) - h(x) \geq p(x_1, x) - p(x, x) \quad (> 0).$$

We construct a sequence $\{x_n\}$ in Xsuch that for each n, if

$$c_n = \inf\{h(x): h(x_n) - h(x) \geq p(x_n, x) - p(x, x) > 0\}, \tag{2.1}$$

then $x_{n+1} \in X$ satisfies

$$h(x_n) - h(x_{n+1}) \geq p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1})$$

and $h(x_{n+1}) < c_n + \frac{1}{n}$. (2.2)

If some x_n is a p-h point we are done; then $x_{n+1} = x_n$ or else (2.1) shows $\{h(x_n)\}$ is monotonic decreasing. Suppose $m > n$. Then we have

$$\begin{aligned}
&h(x_n) - h(x_m)\\
&= \big(h(x_n) - h(x_{n+1})\big) + \big(h(x_{n+1}) - h(x_{n+2})\big) + \cdots + \big(h(x_{m-1}) - h(x_m)\big)\\
&\geq \big(p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1})\big) + \big(p(x_{n+1}, x_{n+2}) - p(x_{n+2}, x_{n+2})\big)\\
&\quad + \cdots + (p(x_{m-1}, x_m) - p(x_m, x_m))\\
&\geq \big(p(x_n, x_{n+2}) - p(x_{n+2}, x_{n+2})\big) + \big(p(x_{n+2}, x_{n+3}) - p(x_{n+3}, x_{n+3})\big)\\
&\quad + \cdots + (p(x_{m-1}, x_m) - p(x_m, x_m))\\
&\geq \big(p(x_n, x_{n+3}) - p(x_{n+3}, x_{n+3})\big) + \cdots + (p(x_{m-1}, x_m) - p(x_m, x_m))\\
&\geq p(x_n, x_m) - p(x_m, x_m) > 0.
\end{aligned}$$

Thus

$$h(x_n) - h(x_m) \geq p(x_n, x_m) - p(x_m, x_m). \tag{2.3}$$

As h is bounded and $\{h(x_n)\}$ is monotonic decreasing, $\{h(x_n)\}$ is convergent and (2.3) says $\{x_n\}$ is p-convergent in the partial metric space (X, p).

Let $p\text{-}\lim_{n\to\infty} x_n = x_0 \in X$.

Then we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0).$$

Now

$$h(x_n) - h(x_0) \geq p(x_n, x_0) - p(x_0, x_0). \tag{2.4}$$

This is true for all n.
Otherwise for some n, we have
$h(x_n) - h(x_0) < p(x_n, x_0) - p(x_0, x_0)$.
Choose $\epsilon > 0$. Then

$$h(x_n) - h(x_0) < p(x_n, x_0) - p(x_0, x_0) - \epsilon.$$

Let $\epsilon_1 = p(x_n, x_0) - p(x_0, x_0) - \epsilon - h(x_n) + h(x_0)(> 0)$. We apply p-l.s.c. of h at x_0 to get

$$h(x_0) - \epsilon_1 < h(x), \quad \text{for all } x \in p\text{-}B_\delta(x_0) \text{ for some } \delta > 0,$$

$$\text{or,} h(x_0) - \{p(x_n, x_0) - p(x_0, x_0) - \epsilon - h(x_n) + h(x_0)\} < h(x),$$

or, $h(x_n) - h(x) < p(x_n, x_0) - p(x_0, x_0) - \epsilon$.
As $\{x_n\} \to x_0$, for large m, $x_m \in p\text{-}B_\delta(x_0)$, so

$$h(x_n) - h(x_m) < p(x_n, x_0) - p(x_0, x_0) - \epsilon$$

$$< p(x_n, x_m) + p(x_m, x_0) - p(x_m, x_m) - p(x_0, x_0) - \epsilon.$$

Taking $\delta = \epsilon$, we have

$$h(x_n) - h(x_m) < p(x_n, x_m) - p(x_m, x_m).$$

This contradicts (2.3) above. Therefore, for all n, (2.4) holds.

We now claim that x_0 is a p-$h$point for h. Otherwise, for some x,

$$h(x_0) - h(x) \geq p(x_0, x) - p(x, x) > 0. (2.5)$$

Replacing n by $n + 1$ in (2.4), we find

$$h(x_{n+1}) - h(x_0) \geq p(x_{n+1}, x_0) - p(x_0, x_0).$$

Therefore

$$h(x) \leq h(x) + h(x_{n+1}) - h(x_0)$$

$= h(x_{n+1}) + h(x) - h(x_0)$
$< c_n + \frac{1}{n} + h(x) - h(x_0)$.
As $h(x) - h(x_0)$ is negative from (2.5), taking large n, we have $h(x) < c_n$,which contradicts (2.1). Hence the proof is done.

Now as we look for the converse of Theorem 2.3, we have Theorem 2.5 below in this connection.

Definition 2.4. Let (X, p) be a partial metric space. A function $h : (X, p) \to \mathbb{R}$ is said to be uniformly continuous on X if given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|h(x) - h(y)| < \epsilon$$

whenever $x, y \in X$ and $p(x, y) - \min\{p(x, x),\ p(y, y)\} < \delta$.

Theorem 2.5. Let (X, p) be a partial metric space which is not complete. Then there exists a uniformly continuous function $h : (X, p) \to \mathbb{R}$ which isbounded below and has no p-h point in X.
Proof. Suppose the partial metric space (X, p) is not complete and let $\{x_n\}$ be a p-Cauchy sequence in X which is not p-convergent. Then $\lim_{m,n\to\infty} p(x_m, x_n)$ is finite.
Let$x \in X$ such that$p(x, x_n) \neq p(x_n, x_n)$. We consider the sequence $\{2p(x, x_n) - 2p(x_n, x_n)\}$ in $\mathbb{R}$.
Since $p(x, x_n) \leq p(x, x_m) + p(x_m, x_n) - p(x_m, x_m)$, we have
$\{p(x, x_n) - p(x_n, x_n)\} - \{p(x, x_m) - p(x_m, x_m)\} \leq p(x_m, x_n) - p(x_n, x_n) \to 0$

as $n \to \infty$, which shows that $\{2p(x, x_n) - 2p(x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Let $h(x)$ be its limit. Then $h(x) > 0$ so that h is bounded below.
Let $x_0 \in X$. Then

$$|h(x_0) - h(x)| = \left|\lim_{n\to\infty}[2p(x_0, x_n) - 2p(x, x_n)]\right|$$
$$= 2[p(x_0, x) - p(x, x)]$$
$$\leq 2[p(x_0, x) - \min\{p(x_0, x_0), p(x, x)\}].$$

Hence h is uniformly continuous on X. Also,

$$h(x_0) + h(x) \geq 2p(x_0, x) - \lim_{n\to\infty} 2p(x_n, x_n).$$

So

$$h(x_0) - h(x) \geq p(x_0, x) + \frac{1}{2}[h(x_0) - 3h(x)] - \lim_{n\to\infty} p(x_n, x_n).$$

Now as $x = x_m$, we have

$$h(x_0) - h(x) \geq p(x_0, x) - p(x_m, x_m) + [p(x_m, x_m) - \lim_{n\to\infty} p(x_n, x_n)] + \frac{1}{2}[h(x_0) - 3h(x)]$$

so that as m becomes large, we get $h(x_m) \to 0$ as $m \to \infty$. Hence

$$h(x_0) - h(x) \geq p(x_0, x) - p(x, x)$$

when $x = x_m$, m is large, which further implies that x_0 is not p-h point.

3. Application in fixed point theory

Let (X, p) be a partial metric space and h be as given. Define $\ll$ on X by the rule that for $x, y \in X$,

$x \ll y$ if and only if $h(y) - h(x) \geq p(x, y) - p(x, x) > 0$.

Then $x \ll y$ relation orders X. The relation is transitive and antisymmetric.

Definition 3.1. A point $x_0 \in X$ is said to be a minimal point with respect to $\ll$ if and only if $x \ll x_0$ implies $x = x_0$.

Theorem 3.2. A point x_0 in X is p-h point for h if and only if x_0 is a minimal point with respect to $\ll$.
Proof: If a point $x_0 \in X$ is a p-h point for h, then

$$h(x_0) - h(x) < p(x_0, x) - p(x, x)$$

for all $x \in X$ with $x \neq x_0$. This gives $x \ll x_0$ only if $x = x_0$. Therefore x_0 is a minimal point with respect to $\ll$.
Conversely, let x_0 be a minimal point with respect to $\ll$, so x $\ll x_0$ implies $x = x_0$, i.e., if $x \neq x_0$, then

$$h(x_0) - h(x) < p(x_0, x) - p(x, x)$$

for all $x \in X$ with $x \neq x_0$. This implies x_0 is a p-h point for h.

Theorem 3.3. Given a function $f : X \to X$, it may be possible to take a partial metric p and a function h so that $\ll$ has the property $f(x) \ll x$. Then any p-h point for h is a fixed point of f.
Proof. Let x_0 be a p-h point for h, then by Theorem 3.2, x_0 is a minimal point with respect to $\ll$. Again by hypothesis $f(x_0) \ll x_0$. It then follows that $f(x_0) = x_0$, a fixed point of f.

4. Some applications of p-$h$points in a metric space

Theorem 4.1. (Kannan Fixed Point Theorem)
Let (X, d) be a complete metric space and $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$ where $0 \leq \alpha < \frac{1}{2}$, and let $d(x, f(x))$ be a l.s.c. function, then f has a fixed point in X.

Proof. We take
$h(x) = \frac{1-\alpha}{1-2\alpha} d(x, f(x))$.
Then $h : X \to \mathbb{R}$ is al.s.c. function being bounded below. In view of Theorem 2.3, we find that there is a d-h point in X.
Now

$$h(x) - h(f(x)) = \frac{1-\alpha}{1-2\alpha}[d(x, f(x)) - d(f(x), f^2(x))].$$

Again,

$$d(f(x), f^2(x)) \leq \alpha[d(x, f(x)) + d(f(x), f^2(x))]$$

which implies

$$d(f(x), f^2(x)) \leq \frac{\alpha}{1-\alpha} d(x, f(x)).$$

Hence

$$\begin{aligned} h(x) - h(f(x)) &\geq \frac{1-\alpha}{1-2\alpha}[d(x, f(x)) - \frac{\alpha}{1-\alpha} d(x, f(x))] \\ &= \left(\frac{1-\alpha}{1-2\alpha}\right)\left(\frac{1-\alpha-\alpha}{1-\alpha}\right) d(x, f(x)) \\ &\geq d(x, f(x)). \end{aligned}$$

Therefore $f(x) \ll x$. Now we apply Theorem 3.3 to conclude that f has a fixed point in X.

Theorem 4.2. (B. Fisher Theorem)
Let (X, d) be a complete metric space and $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq \alpha[d(x, f(y)) + d(y, f(x))]$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{2}$ and let $d(x, f(x))$ be a l.s.c. function, then f has a fixed point in X.

Proof: Here also we take

$$h(x) = \frac{1-\alpha}{1-2\alpha} d(x, f(x)).$$

Then

$$h(x) - h(f(x)) = \frac{1-\alpha}{1-2\alpha}[d(x, f(x)) - d(f(x), f^2(x))].$$

Now

$$\begin{aligned} d(f(x), f^2(x)) &\leq \alpha[d(x, f^2(x)) + d(f(x), f(x))] \\ &\leq \alpha[d(x, f(x)) + d(f(x), f^2(x))] \end{aligned}$$

so that

$$d(f(x), f^2(x)) \leq \frac{\alpha}{1-\alpha} d(x, f(x)).$$

Hence

$$h(x) - h\big(f(x)\big) \geq d\big(x, f(x)\big).$$

Thus $f(x) \ll x$. Since by Theorem 2.3, we get a d-h of h in X, we find that there exists a fixed point of f in X.

Theorem 4.3. Let (X, d) be a complete metric space and $f: X \rightarrow X$ satisfies

$$d\big(f(x), f(y)\big) \leq \alpha\big[d\big(x, f(x)\big) + d\big(y, f(y)\big)\big] + \beta d(x, y)$$
$$+ \gamma \max\{d\big(x, f(y)\big), d\big(y, f(x)\big)\}$$

for all $x \in X$ where $\alpha, \beta, \gamma \geq 0$, $2\alpha + \beta + 2\gamma < 1$, and let $d(x, f(x))$ be a l.s.c. function, then f has a fixed point in X.

Proof. Here we take $h(x) = a\, d\big(x, f(x)\big)$, for all $x \in X$, where $a = \frac{1-\alpha-\gamma}{1-2\alpha-\beta-2\gamma}$.

Then

$$h(x) - h\big(f(x)\big) = a\,\big[d\big(x, f(x)\big) - d\big(f(x), f^2(x)\big)\big].$$

Now

$$d\big(f(x), f^2(x)\big) \leq \alpha\big[d\big(x, f(x)\big) + d\big(f(x), f^2(x)\big)\big] + \beta d\big(x, f(x)\big)$$
$$+\gamma \max\{\, d(x, f^2(x)), d\big(f(x), f(x)\big)\}$$
$$\leq \alpha\big[d\big(x, f(x)\big) + d\big(f(x), f^2(x)\big)\big] + \beta d\big(x, f(x)\big) + \gamma d(x, f^2(x))$$
$$\leq (\alpha + \beta + \gamma) d\big(x, f(x)\big) + (\alpha + \gamma) d\big(f(x), f^2(x)\big).$$

This gives

$$d\big(f(x), f^2(x)\big) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma} d\big(x, f(x)\big),$$

which implies

$$h(x) - h\big(f(x)\big) \geq a\left[1 - \frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma}\right] d\big(x, f(x)\big) = d\big(x, f(x)\big).$$

Hence $f(x) \ll x$. Then we proceed as before to complete the proof.

5. Conclusions

In this paper, a kind of characterization of completeness property of a partial metric space has been achieved, by using the notion of p-h point relevant to fixed point theory in the space.

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