Annals of Pure and Applied Mathematics
Vol. 16, No. 2, 2018, 437-441
ISSN: 2279-087X (P), 2279-0888(online)
Published on 24 March 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v16n2a21

Annals of **Pure and Applied Mathematics**

On a Ramsey Problem Involving the 3-Pan Graph

Chula Jayawardene

Department of Mathematics, University of Colombo, Sri Lanka
Email: c_jayawardene@yahoo.com

Received 20 February 2018; accepted 21 March 2018

Abstract. Let K_s and $K_{j\times s}$ denote the complete graph on s vertices and the complete multipartite balanced graph having j partite sets (where $j \geq 3$) of size s respectively. For any two graphs say G, H, we say that $K_s \to (H,G)$, if for any red/blue coloring of K_s, given by $K_s = H_R \oplus H_B$, there exists a red copy of a H in H_R or a blue copy G in H_B. In accordance with the same notation, we also say that $K_{j\times s} \to (H,G)$, if for any red/blue coloring of $K_{j\times s}$, given by $K_{j\times s} = H_R \oplus H_B$, there exists a red copy of a H in H_R or a blue copy G in H_B. The balanced multipartite Ramsey number $m_j(G,H)$ is defined as the smallest positive number s such that that $K_{j\times s} \to (H,G)$. There are 11 non-isomorphic graphs G on 4 vertices, out of which 5 graphs G are connected and the others are disconnected. In this paper we exhaustively find $m_j(P,G)$ for all of the 11 non-isomorphic graphs G on 4 vertices where P denotes the 3-pan graph (paw graph) given by $K_{1,3}+e$.

Keywords: Graph theory, Ramsey theory

AMS Mathematics Subject Classification: 05C55, 05D10

1. Introduction

All graphs mentioned in this paper are simple graphs that do not contain loops or parallel edges. The diagonal classical Ramsey number $r(n,n)$, defined as the smallest positive integer t such that $K_t \to (K_n,K_n)$, have been studied in detail and are known for almost all pairs of graphs when $n < 5$. However, not much is known about the exact value of $r(5,5)$ other than that the upper bound is 48 (proved by Vigleik Angeltveit and Brendan D. McKay).

A new branch of the classical Ramsey numbers, namely the size Ramsey multipartite numbers $m_j(H,G)$, were introduced by Van Vuurenet al ([1]) and Baskoro et al([8]), a few decades ago. As of yet, the exact value $m_j(H,G)$, when $|V(G)| < 5$, and $|V(H)| < 5$ are known only for a few pairs of graphs. In this paper we exhaustively find the exact value of $m_j(P,G)$ for all of the 11 non-isomorphic graphs G on 4 vertices, when P is isomorphic to the 3-pan graph.

The summary of our findings is illustrated in the following table.

$m_j(K_{1,3}$+e,$G)$	j = / Graph G	3	4	5	6	7	8	9	Greater than or equal to 10
Row 1	$4K_1$	2	1	1	1	1	1	1	1
Row 2	$P_2 U 2K_1$	2	1	1	1	1	1	1	1
Row 3	$2K_2$	2	2	1	1	1	1	1	1
Row 4	$P_3 U K_1$	2	2	1	1	1	1	1	1
Row 5	P_4	3	2	2	1	1	1	1	1
Row 6	$K_{1,3}$	3	3	2	2	1	1	1	1
Row 7	$C_3 U K_1$	∞	∞	∞	2	1	1	1	1
Row 8	C_4	3	2	2	2	1	1	1	1
Row 9	$K_{1,3} + e$	∞	∞	∞	2	1	1	1	1
Row 10	B_2	∞	∞	∞	2	1	1	1	1
Row 11	K_4	∞	∞	∞	∞	∞	∞	2	1

Table 1: Values of $m_j(K_{1,3}$+e,$G)$.

The next section deals with finding the entries of the above table. Clearly the rows corresponding to row 1, row 2, row 4, row 5, row 6 and row 10 follows from Syafrizal and et al and Jayawardene et al (see [3, 4, 5, 6, 8]).

2. Some useful lemmas on connected subgraphs of K_4

Theorem 1. *If $j \geq 3$, then*

$$m_j(K_{1,3}+e, C_3) = \begin{cases} 1 & j \geq 7 \\ 2 & j = 6 \\ \infty & j \in \{3, 4, 5\} \end{cases}$$

Proof: If $j \geq 7$, since $r(K_{1,3}+\mathrm{e}, C_3) = 7$ (see [2]), we get $m_j(K_{1,3}$+e, $C_3) = 1$.

Consider the graph $K_{6\times 1} = H_R \oplus H_B$, such that H_R equals to a $2K_3$ and H_B equals to a $K_{3,3}$. Then the graph has no red $K_{1,3}$+e and has no blue C_3. Therefore, $m_6(K_{1,3}$+e, $C_3) \geq 2$. Next to show, $m_6(K_{1,3}$+e, $C_3) \leq 2$ consider any red/blue coloring given by $K_{6\times 2} = H_R \oplus H_B$, such that H_R contains no red $K_{1,3}$+eand H_B contains no blue C_3. As $r(C_3, C_3) = 6$ from [2] there is a red C_3, in H_R. Without loss of generality, assume that the red C_3, is induced by say v_{11},v_{21}, v_{31}. Let $S = \{v_{i2} \mid i \in \{4,5,6\}\}$. Since H_R contains no red $K_{1,3}$+e, all edges joining v_{11} to each of the 6 elements in S will be blue. If we consider the red/blue graphs generated by S, as $m_3(K_{1,3}+e, P_2) = 2$, we get that it will contain a blue P_2. But then the vertices of this P_2 together with v_{11} will give us a blue C_3, a contradiction. Hence, $m_6(K_{1,3}$+e, $C_3) \leq 2$. Therefore, $m_6(K_{1,3}$+e, $C_3) = 2$.

Finally, as $m_i(K_{1,3}+e, C_3) \geq m_i(C_3, C_3)$ for all I and $m_i(C_3, C_3) = \infty$ for $j \in \{3, \ldots, 5\}$ (see [5]), we get that $m_i(K_{1,3}+e, C_3) = \infty$ for $j \in \{3, \ldots, 5\}$.

Theorem 2. *If* $j \geq 3$, *then*

$$m_j(K_{1,3} + e, C_4) = \begin{cases} 1 & j \geq 7 \\ 2 & j \in \{4,\ 5,\ 6\} \\ 3 & j = 3 \end{cases}$$

Proof: Let $j \geq 3$. All values of $m_j(C_4, C_3)$ has been found in [5]. This gives us, $m_j(K_{1,3} + x, C_4)$ since $m_j(K_{1,3} + e,\ C_4) = m_j(C_3, C_4)$.

Theorem 3. *If* $j \geq 3$, *then*

$$m_j(K_{1,3} + e, K_{1,3} + e) = \begin{cases} 1 & j \geq 7 \\ 2 & j = 6 \\ \infty & j \in \{3, 4, 5\} \end{cases}$$

Proof: If $j \geq 7$, since $r\left(K_{1,3}\text{+e}, K_{1,3} + e\right) = 7$ (see [2]), we get $m_j(K_{1,3}$+e, $K_{1,3}$+e) = 1.

Next color the graph $K_{6\times1} = H_R \oplus H_B$, such that $H_R = 2K_3$. Then the graph has no red $K_{1,3}$+e and has no blue $K_{1,3}$+e. Therefore, m_6 $(K_{1,3}$, $K_{1,3}$+e) ≥ 2. Next to show, $m_6(K_{1,3}$+e, $K_{1,3}$+e) $\leq$ 2, consider any red/blue coloring given by $K_{6\times2} = H_R \oplus H_B$, such that H_R contains no red $K_{1,3}$+e and H_B contains no blue $K_{1,3}$ +e. As $m_6(C_3, K_{1,3}\text{+e}) = 2$ from [5] there is a red C_3, in H_R. Without loss of generality assume that the red C_3, is induced by say v_{11}, v_{21}, v_{31}. But then if we consider the vertex v_{11} it must be adjacent in blue to all of the vertices of v_{41}, v_{42}, v_{52}, v_{62} as otherwise would result in a red $K_{1,3}$+e. But then all the edges (v_{41}, v_{52}), (v_{41}, v_{62}), (v_{42}, v_{52}), (v_{42}, v_{62}) and $(v_{52},\ v_{62})$ will be forced to be red as otherwise it will result in a blue $K_{1,3}$+e.

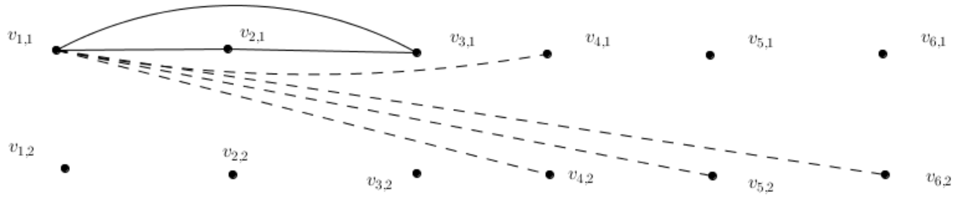

Figure 1: Diagram related to the proof of $m_6(K_{1,3}$+e, $K_{1,3}$+e) ≤ 2

But then the vertex set S ={v_{41},v_{42}, v_{52}, v_{62} } will contain a red $K_{1,3}$+e, a contradiction. Thus, $m_6(K_{1,3}+e, K_{1,3}+e) \leq 2$ Therefore, we get $m_6(K_{1,3}+e, K_{1,3}+e) = 2.$

When $j \in \{3,4,5\}$, $m_3(C_3, K_{1,3}+e) = \infty$ follows from [5]. Therefore, as C_3, is a subgraph $K_{1,3}+e$, it follows that, $m_j(K_{1,3}+e, K_{1,3}+e) = \infty$ for $j = \{3,4,5\}$, as required.

Theorem 4. *If $j \geq 3$, then*

$$m_j(K_{1,3}+e, K_4) = \begin{cases} 1 & j \geq 10 \\ 2 & j = 9 \\ \infty & j \in \{3,\ldots,8\} \end{cases}$$

Proof: If $j \geq 10$, since $r\left(K_{1,3}+e, K_4\right) = 10$ (see [2]), we get $m_j(K_{1,3}+e, K_4) = 1$.

Consider the graph$K_{9\times1} = H_R \oplus H_B$, such that H_R equals to a $3K_3$ and H_B equals to a $K_{3,3,3}$. Then the graph has no red $K_{1,3}$+e and has no blue K_4. Therefore,$m_9(K_{1,3}+e, K_4) \geq 2$. Next to show, $m_9(K_{1,3}+e, K_4) \leq 2$ consider any red/blue coloring given by $K_{9\times2} = H_R \oplus H_B$, such that H_R contains no red $K_{1,3}$+eand H_B contains no blue K_4. As $r(C_3, K_4) = 9$ from [2] there is a red C_3, in H_R. Without loss of generality assume that the red C_3, is induced by say v_{11},v_{21}, v_{31}. Let $S = \{v_{i2} \mid i \in \{2,3,\ldots,8\}\}$. Since H_R contains no red $K_{1,3}$+e, all edges joining v_{11} to each of the 7 elements in S will be blue.If we consider the red/blue graphs generated by S, as $r(K_{1,3}+e, C_3) = 7$, we get that it will contain a blue C_3. But then the vertices of this C_3 together with v_{11} will give us a blue K_4, a contradiction. Hence, $m_9(K_{1,3}+e, K_4) \leq 2$. Therefore, $m_9(K_{1,3}+e, K_4) = 2$.

Finally, as $m_i(K_{1,3}+e, K_4) \geq m_i(C_3, K_4)$ for all i, and $m_i(C_3, K_4) = \infty$ for $j \in \{3,\ldots,8\}$ (see [5]), we get that $m_i(K_{1,3}+e, K_4) = \infty$ for $j \in \{3,\ldots,8\}$.

3. Size Ramsey numbers $m_j(P_4, G)$ when G is disconnected graph on 4 vertices

We have already dealt with all cases excluding $G = 2K_2$. We will deal with this in the following theorem.

Theorem 5. *If $j \geq 3$, then*

$$m_j(K_{1,3}+e, 2K_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \text{if } j \geq 5 \end{cases}$$

Proof: Consider the coloring of $K_{4\times1} = H_R \oplus H_B$, generated by $H_R = K_3$. Then, $K_{4\times1}$ has no red $K_{1,3}$+eor a blue $2K_2$. Therefore, we obtain that $m_4(K_{1,3}+e, 2K_2) \geq 2.$

To show $m_3(K_{1,3}+e,2K_2)\le 2,$ consider any red/blue coloring given by $K_{3\times 2} = H_R \oplus H_B$, such that H_R contains no red $K_{1,3}$+eand H_B contains no blue$2K_2$. Since H_R contains no red $K_{1,3}+e$ without loss of generality we may assume that there is at least one blue edge in $K_{3\times 2}$ say (v_{11},v_{21}). Next as there is no blue $2K_2$ all edges not adjacent to (v_{11},v_{21}) in $K_{3\times 2}$ must be red. Thus, in particular (v_{12},v_{22}), (v_{12},v_{31}), (v_{12},v_{32}) and (v_{22},v_{31}) must be red edges. Thus, we get a red. $K_{1,3}+e,$ a contradiction. That is, $m_3(K_{1,3}+e,2K_2)\le 2.$ Therefore, $m_3(K_{1,3},2K_2)=2$ and $m_4(K_{1,3},2K_2)=2.$ Finally, $m_j(K_{1,3}+e,2K_2)=1$ when $j\ge 5,$ as $r(K_{1,3}+e,2K_2) = 5$ (see [2]).

REFERENCES

1. A.P.Burger and J.H.Van Vuuren, Ramsey numbers in complete balanced multipartite graphs. Part II: size numbers, *Discrete Math.*, 283 (2004) 45-49.
2. V.Chv´atal and F.Harary, Generalized Ramsey theory for graphs, III. Small off diagonal numbers, *Pacific Journal of Mathematics*, 41(2) (1972) 335-345.
3. M.Christou, S.Iliopoulos and M.Miller, Bipartite Ramsey numbers involving stars, stripes and trees, *Electronic Journal of Graph Theory and Applications*, 1(2) (2013) 89-99.
4. C.J.Jayawardene and L.Samerasekara, Size Multipartite Ramsey numbers for K_4-*e*versus all graphs *G* up to 4 vertices, *Annals of Pure and Applied Mathematics*, 13(1) (2017) 9-26.
5. C.J.Jayawardene and L.Samerasekara, Size Ramsey numbers for C_3 versus all graphs *G* up to 4 vertices, *National Science Foundation*, 45(11) (2017) 67-72.
6. V.Kavitha and R.Govindarajan, A study on Ramsey numbers and its bounds, *Annals of Pure and Applied Mathematics*, 8(2) (2014) 227-236.
7. M.S.Sunitha and S.Mathew, Fuzzy graph theory: a survey, *Annals of Pure and Applied Mathematics*, 4(1) (2013) 92-110.
8. S.Sy, E.T.Baskoro and S.Uttunggadewa, The size multipartite Ramsey number for paths, *Journal Combin. Math. Combin. Comput.*, 55 (2005) 103-107.