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Glivenko Congruence on a Nearlattice Related to Semi Prime Ideals

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Abstract. In this paper authors have proved some results on Glivenko congruence R with respect to Semi prime ideal J in a nearlattice S. They showed that the quotient nearlattice $\frac{S}{R}$ is distributive if and only if J is semiprime. Moreover, they have included a prime separation theorem for semiprime ideals. At the end some results on $A^{\perp}$ and A^{0} for a 0-distributive nearlattice are given. Finally they have included a characterization of distributive nearlattices with the help of Separation theorems by using semiprime ideals.

Keywords: 0-distributive nearlattice, Semi prime ideal, Glivenko congruence, Quotient nearlattice.

AMS Mathematics Subject Classification (2010): 06A12, 06A99, 06B10

1. Introduction

Varlet [8] first introduced the concept of 0-distributive lattices. Then many authors [2, 6] studied them for lattices and semilattices. On the other hand, [4] studied the 0-distributive directed above meet semilattices extensively and discussed different properties of these semilattices by a number of characterizations. Recently [9, 10] have studied them for nearlattices. Again [5] have proved several interesting results on 0-distributive nearlattices.

A *nearlattice* is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the *upper bound property.*

By [9], a nearlattice S with 0 is called 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$ whenever $b \vee c$ exists. [7] Introduced the concept of semiprime ideals in a lattice. Then [1] studied these ideals elaborately and established many interesting results. They also extend the Prime Separation Theorem for 0-distributive lattices, which give a flavour of Separation Theorem for non-distributive lattices. Then [11] extended the concept for nearlattices. An ideal J of a nearlattice S is called a semiprime ideal if for all $a,b,c\in S$ with $a\wedge b\in J$ and $a\wedge c\in J$ imply $a\wedge(b\vee c)\in J$ whenever $b\vee c$ exists. Hence a nearlattice S with 0 is called 0-distributive if $(0]$ is a semiprime ideal of S. Let $A\subseteq S$ and J be an ideal of S. We define $A^{\perp J}=\{x\in S: x\wedge a\in J \textit{ for all } a\in A\}$. This is clearly a down set containing J. By [11, Theorem 5], we know that when J is semiprime then $A^{\perp J}$ is in fact a semiprime ideal. $A^{\perp J}$ called an annihilator of A relative to J.

For an ideal J of a nearlattice S, define a relation θ on S by $a\equiv b(\theta)$ if and only if $(a]^{\perp J}=(b]^{\perp J}$. In otherwords, $a\equiv b(\theta)$ is equivalent to " for each $x\in S$, $a\wedge x\in J$ if and only if $b\wedge x\in J$."

In this paper, we will show that this is a congruence on the nearlattice S when J is semiprime. We call it as Glivenko congruence. Recently Glivenko congruence have been studied by [12]. In this paper, we extend several results of [12] and then establish some new results.

2. Main result

Proposition 2.1. Let J be a semiprime ideal in a nearlattice S. Define a relation R on S by $x\equiv y(R)$ if and only if $\{x\}^{\perp J}=\{y\}^{\perp J}$. Then R is a nearlattice congruence on S.

Proof: Clearly R is an equivalence relation on S. Now let $x\equiv y(R)$ and $t\in S$. Then $\{x\}^{\perp J}=\{y\}^{\perp J}$. Suppose $a\in\{x\wedge t\}^{\perp J}$. Then $a\wedge x\wedge t\in J$ which implies $a\wedge t\in\{x\}^{\perp J}=\{y\}^{\perp J}$. Thus, $a\wedge y\wedge t\in J$ and so $a\in\{y\wedge t\}^{\perp J}$. Therefore $\{x\wedge t\}^{\perp J}\subseteq\{y\wedge t\}^{\perp J}$. Similarly, $\{y\wedge t\}^{\perp J}\subseteq\{x\wedge t\}^{\perp J}$ and so $\{x\wedge t\}^{\perp J}=\{y\wedge t\}^{\perp J}$. Hence $x\wedge t\equiv y\wedge t(R)$. Now let $x\equiv y(R)$ and $x\vee t,\ y\vee t$ exist for some $t\in S$. Let $a\in\{x\vee t\}^{\perp J}$. Then $a\wedge(x\vee t)\in J$ and so $a\wedge x, a\wedge t\in J$. This implies $a\wedge y, a\wedge t\in J$ as $\{x\}^{\perp J}=\{y\}^{\perp J}$. Therefore $a\wedge(y\vee t)\in J$ as J is semiprime. It follows that R is a nearlattice congruence on S.●

Remarks: Let S be a nearlattice and Θ a congruence on S. We denote the quotient nearlattice of S modulo Θ by $\frac{S}{\Theta}$. If $\frac{S}{\Theta}$ has a zero element $[0]$, then $[0]$ is called the kernel of Θ. Clearly $[0]$ is then an ideal of S. Notice that we do not require S itself to have a zero element. If J is an ideal of S, we shall say that J is the kernel of a homomorphism if there exists a congruence Θ on S such that J is the kernel of Θ. Thus an ideal J is a kernel provided J is a complete congruence class for some congruence Θ on S. Then for every $x \in S$ and any $j \in J$, $x \geq x \wedge j$ implies $\frac{[x]}{\Theta} \geq J$ in $\frac{S}{\Theta}$. Hence J is the zero element of $\frac{S}{\Theta}$.

Theorem 2.2. Let S be a nearlattice and J be an ideal of S. Then the following conditions are equivalent.

(i) J is semiprime.

(ii) J is the kernel of some homomorphism of S onto a distributive neralattice with 0.

(iii) J is the kernel of some homomorphism of S onto a 0-distributive nearlattice.

Proof: $(i) \Rightarrow (ii)$ Consider the elements $[x], [y], [z]$ in $\frac{S}{R}$ such that $y \vee z$ exists where R is the Glivenko congruence. Let $s \equiv x \wedge (y \vee z)(R)$. Then $\{s\}^{\perp J} = \{x \wedge (y \vee z)\}^{\perp J}$. Suppose $t \in \{s\}^{\perp J}$. Then $t \wedge (x \wedge (y \vee z)) \in J$, hence $t \wedge x \in \{y \vee z\}^{\perp J} = \{y\}^{\perp J} \cap \{z\}^{\perp J}$.

Therefore, $t \in \{x \wedge y\}^{\perp J} \cap \{x \wedge z\}^{\perp J} = \{(x \wedge y) \vee (x \wedge z)\}^{\perp J}$. Thus $\{s\}^{\perp J} \subseteq \{(x \wedge y) \vee (x \wedge z)\}^{\perp J}$, equivalently, $\frac{[s]}{R} \leq \frac{[(x \wedge y) \vee (x \wedge z)]}{R}$, hence $\frac{[x]}{R} \wedge \left(\frac{[y]}{R} \vee \frac{[z]}{R}\right) \leq \left(\frac{[x]}{R} \wedge \frac{[y]}{R}\right) \vee \left(\frac{[x]}{R} \wedge \frac{[z]}{R}\right)$. Since the reverse inequality is trivial, so $\frac{S}{R}$ is a distributive nearlattice.

Furthermore, for any $i, j \in J$ clearly $i \equiv j(R)$. Moreover, for any $i \in J$, $i \equiv a(R)$ implies $\{a\}^{\perp J} = \{i\}^{\perp J} = S$. This implies $a \in J$. Thus J is a complete congruence class modulo R. That is, J is the kernel of R and so (ii) holds.

$(ii) \Rightarrow (iii)$ By (ii) $J = ker\,\Theta$ for some congruence Θ on S and $\frac{S}{\Theta}$ is a distributive nearlattice. Since every distributive nearlattice S with 0 is 0-distributive, so $\frac{S}{\Theta}$ is 0-distributive and so (iii) holds.

$(iii) \Rightarrow (i)$ Let Θ be a congruence on S for which J is the zero element of the 0-distributve nearlattice $\frac{S}{\Theta}$. Let $x \wedge y \in J$ and $x \wedge z \in J$ such that $y \vee z$ exists. This implies $\frac{[x]}{\Theta} \wedge \frac{[y]}{\Theta} = \frac{[x \wedge y]}{\Theta} = J = \frac{[y \wedge z]}{\Theta} = [y] \wedge [z]$. Since $\frac{S}{\Theta}$ is 0-distributive, it follows that $\frac{[x]}{\Theta} \wedge \frac{([y] \vee [z])}{\Theta} = J$. That is, $\frac{[x \wedge (y \vee z)]}{\Theta} = J$ and so $x \wedge (y \vee z) \in J$. Therefore J is semiprime. ●

Now we give a separation theorem for semiprime ideals.

Theorem 2.3. Let J be a semiprime ideal of a nearlattice S and F be a filter of S disjoint to J. Then there exists a prime ideal $P \supseteq J$ such that $P \cap F = \phi$.

Proof: Define a relation R on S by $x \equiv y(R)$ if and only if $\{x\}^{\perp J} = \{y\}^{\perp J}$. Then by Proposition 2.1 and Theorem 2.2, R is a nearlattice congruence and the quotient nearlattice $\frac{S}{R}$ is distributive. Since $F \cap J = \phi$, so $\frac{F}{R}$ is a proper filter of $\frac{S}{R}$. It follows now from the prime separation theorem for distributive nearlattice [3] that there exist a prime ideal $\frac{P}{R}$ of $\frac{S}{R}$ disjoint to $\frac{F}{R}$. Then clearly, $P = h^{-1}\left(\frac{P}{R}\right)$ is a prime ideal of S containing J and disjoint from F, where h is the canonical homomorphism of S onto $\frac{S}{\Theta}$. ●

By [11, Theorem 5] we know that for any subset A of a 0-distributive nearlattice S, $A^{\perp} = \{x \in S \mid x \wedge a = 0 \ for\ all\ a \in A\}$ is a semiprime ideal. We also define $A^{0} = \{x \in S \mid x \wedge a = 0 \ for\ \ some\ a \in A\}$.

Lemma 2.4. For a meet subsemilattice A of a 0-distributive nearlattice S, A^0 is a semiprime ideal.

Proof: By [10, Theorem5], A^0 is an ideal. Now let $x \wedge y \in A^0$ and $x \wedge z \in A^0$ for some $x, y, z \in S$ with $y \vee z$ exists. Then $x \wedge y \wedge a = 0$ and $x \wedge z \wedge b = 0$ for some $a, b \in A$. Since A is a meet subsemilattice, so $a \wedge b \in A$.

Now $x \wedge y \wedge a \wedge b = 0 = x \wedge z \wedge a \wedge b$ imply $x \wedge a \wedge b \wedge (y \vee z) = 0$ as S is a 0-distributive nearlattice. Thus $x \wedge (y \vee z) \in A^0$ and so A^0 is semiprime. ●

Thus we have the following corollaries.

Corollary 2.5. Let A be a non-empty subset and F be a filter of a 0-distributive nearlattice S such that $A^{\perp} \cap F = \phi$. Then there exists a prime ideal P containing $A^{\perp}$ such that $P \cap F = \phi$. ●

Corollary 2.6. Let A be a meet subsemilattice and F be a filter of a 0-distributive nearlattice S such that $A^0 \cap F = \phi$ Then there exists a prime ideal P containing A^0 and $P \cap F = \phi$.●

Theorem 2.7. Let J be a semiprime ideal of a nearlattice S and suppose that for some $a,b \in S$, $a \wedge b \in J$. Then there exist semiprime ideals A and B (possibly improper) such that $a \in A$, $b \in B$ and $J = A \cap B$.

Proof: If $a \in J$, then by choosing $A = J$ and $B = S$, the theorem trivially holds. So assume hence forth that neither a nor b is in J. Now define the relation R on S by $x \equiv y(R)$ if and only if $\{x\}^{\perp_J} = \{y\}^{\perp_J}$. Since J is semiprime , so by theorem 2.2, R is a nearlattice congruence and $\frac{S}{R}$ is a distributive nearlattice. Let $h : S \to \frac{S}{R}$ be the canonical homomorphism with kernel J. Put $S' = \frac{S}{R}$. Thus S' is a distributive nearlattice with $0' = J$. Hence $a' \wedge b' = 0'$, where $a' = h(a)$ and $b' = h(b)$. By hypothesis $a \notin J$, $b \notin J$, hence $a' \neq 0' \neq b'$. Choose the ideals $A' = (a' \vee b'] \cap (b']^*$ and $B' = (a' \vee b'] \cap (a']^*$ in $\frac{S}{R}$. Since $a' \wedge b' = 0'$ it follows that $a' \in A'$ and $b' \in B'$. Clearly $A' \cap B' = (0']$. Putting $A = h^{-1}A'$ and $B = h^{-1}B'$ yields the semiprime ideals A and B.●

Theorem 2.8. A nearlattice S is distributive if and only if for every ideal I and a filter F of S for which $I \cap F = \phi$ there exists a semiprime ideal $J \supseteq I$ such that $J \cap F = \phi$.

Proof: Suppose S is distributive. Then by [3, Theorem 2.6], there exists a prime ideal $P \supseteq I$ such that $P \cap F = \phi$. Since every prime ideal is semiprime, so choosing $J = P$ we get the result.

Conversely, suppose the condition holds. If S is not distributive. Then there exist $x,y,z \in S$ with $y \vee z$ exists such that $x \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z)$. Consider $I = ((x \wedge y) \vee (x \wedge z)]$ and $F = [x \wedge (y \vee z))$. Clearly $I \cap F = \phi$. Then by the given condition, there exists a semiprime ideal $J \supseteq I$ such that $J \cap F = \phi$. Now $x \wedge y \in J$ and $x \wedge z \in J$. Since J is semiprime , so $x \wedge (y \vee z) \in J$. This implies $J \cap F \neq \phi$, which gives a contradiction. Hence S must be distributive.●

Finally we include the following Separation Theorem.

Theorem 2.9. Let F be a proper filter of a near lattice S with 0. Then the following conditions are equivalent.

(i) S is 0-distributive.

(ii) For each finite subset A of S with $F \cap A^{\perp} = \phi$, there exists prime filter $Q \supseteq F$ such that $Q \cap A^{\perp} = \phi$.

Proof: $(i) \Rightarrow (ii)$. Since S is 0-distributive, so by [11, Theorem5], $A^{\perp}$ is a semiprime ideal. Thus by corollary 2.5 (ii) holds.

$(ii) \Rightarrow (i)$. Conversely, let (ii) holds but S is not 0-distributive. Then there exists $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists but $a \wedge (b \vee c) \neq 0$. Set $F = \{ x \in S : x \geq a \wedge (b \vee c) \}$. Then F is a proper filter as $0 \notin F$. Also $a \in F$ and $b \vee c \in F$. But for any $x \in F, a \wedge x \geq a \wedge (b \vee c) \neq 0$. implies $x \notin A^{\perp}$. Therefore, $F \cap A^{\perp} = \phi$. Hence by (ii), there exists a prime filter $Q \supseteq F$ such that $Q \cap A^{\perp} = \phi$. Then $b \vee c \notin S - Q$. This implies either $b \notin S - Q$ or $c \notin S - Q$ as $S - Q$ is a prime ideal. Hence either $b \in Q$ or $c \in Q$. Therefore, either $a \wedge b \in Q$ or $a \wedge c \in Q$. Which implies $0 \in Q$ and this gives a contradiction that Q is a prime filter. Hence S must be 0-distributive. ●

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3. Conclusion

Glivenko congruence is an important topic distributive lattices. This concept has been extended for 0-distributive lattices. In this paper we have generalized the concept for 0-distributive nearlattices, which are of course non distributive nearlattices. In future, extending the results of this paper futher research can be down for general non distributive nearlattices with 0.

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