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On Generalized Bounded Mocanu Variation Related to q-Derivative and Conic Regions

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Abstract. In this paper, we introduce a class of normalized analytic functions defined in the open unit disc. We study the unified class of functions with bounded Mocanu variation which map the open unit disk onto conic regions using q-derivative. We invistigate several Interesting mapping properties, certain inclusion results and generalized type of q-Bernardi integral operator for this class.

1. Introduction
Let A denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\tag{1.1}$$

which are analytic in the open unit disk $\mathsf{U} = \{z : z \in \mathsf{C} \text{ and } |z| < 1\}$,

and S denote the subclass of A consisting of all function which are univalent in U.

Noor and Malik [14] introduced the classes of generalized bounded Mocanu variation associated with conic domains which generalize many classes studied in literature by Rönning, Goodman, Kanas, Shams and Wisniowska.

The aim of our result is to generalize the class of functions with bounded Mocanu variation which map the open unit disk onto conic domains. Interesting properties of these functions are investigated.

Let $f, g \in \mathsf{A}$. If there exists a function $\phi(z)$ with $\phi(0) = 0$, $|\phi(z)| < 1$, $z \in \mathsf{U}$, such that $f(z) = g(\phi(z))$, then we say $f(z)$ is subordinate to $g(z)$ and we denote this by $f \prec g$. Jackson[8] initiated q-calculus and developed the concept of the q-integral and q-derivative.

For a function $f \in \mathsf{S}$ given by (1.1) and $0 < q < 1$, the q-derivative of f is defined by

$$\partial_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{1.2}$$

Equivalently (1.2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Note that as $q \to 1$, $[n]_q \to n$. For a function $f(z) = z^n$, we can observe that

$$\partial_q f(z) = \partial_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}.$$

Then

$$\lim_{q \to 1} \partial_q f(z) = \lim_{q \to 1} [n]_q z^{n-1} = n z^{n-1} = f'(z),$$

where $f'(z)$ is the ordinary derivative. For more properties of see[3, 5]. As a right inverse, Jackson[9] presented the q-integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k),$$

provided that the series converges. For a function $f(z) = z^k$, we note that

$$\int_0^z f(t) d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \qquad (k \neq -1)$$

and

$$\lim_{q \to 1} \int_0^z f(t) d_q t = \lim_{q \to 1} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z f(t) dt,$$

where $\int_0^z f(t) dt$ is the ordinary integral.

Under the hypothesis of the definition of qâˆ'difference operator, then we have the following rules.

(i) $\partial_q(af(z) \pm bg(z)) = a\partial_q f(z) \pm b\partial_q g(z)$, where a and b any real (or complex) constants

(ii) $\partial_q(f(z)g(z)) = g(qz)\partial_q f(z) + f(z)\partial_q g(z) = f(z)\partial_q g(z) + \partial_q f(z)g(qz)$

(iii) $\partial_q \left(\dfrac{f(z)}{g(z)} \right) = \dfrac{g(z)\partial_q f(z) - f(z)\partial_q g(z)}{g(qz)g(z)}$

Denote by $P(q)$ $(0 < q < 1)$ the family of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in the open unit disc U and satisfying

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$$| p(z) - \frac{1}{1-q} | < \frac{1}{1-q}, \quad z \in \mathsf{U}, \quad q \in (0,1).$$

Definition 1.1. *A function* $f(z) \in \mathsf{A}$ *is said to be in the class* $k - UB_q(\alpha, \beta, \gamma)$, $k \geq 0$, *if and only if, for* $\alpha \geq 0$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, $0 < q < 1$,

$$Re\{J_q(\alpha, \beta, \gamma, f(z))\} > k \,| J_q(\alpha, \beta, \gamma, f(z)) - 1 |. \tag{1.3}$$

where

$$J_q(\alpha, \beta, \gamma, f(z)) = \frac{1-\alpha}{1-\beta}\left\{ \frac{z \partial_q f(z)}{f(z)} - \beta \right\} + \frac{\alpha}{1-\gamma}\left\{ 1 - \gamma + \frac{qz \partial_q^2 f(z)}{\partial_q f(z)} \right\}. \tag{1.4}$$

We note that:

    (i) $\lim_{q \to 1^-} k - UB_q(\alpha, \beta, \gamma) = k - UB(\alpha, \beta, \gamma)$, (Noor and Malik [14]);

    (ii) $\lim_{q \to 1^-} k - UB_q(0,0,\gamma) = k - ST$ and $\lim_{q \to 1^-} k - UB_q(1,\beta,0) = k - UCV$, (Kanas and Wisniowska [11, 12]);

    (iii) $\lim_{q \to 1^-} k - UB_q(0, \beta, \gamma) = SD(k, \beta)$ and

$\lim_{q \to 1^-} k - UB_q(1, \beta, \gamma) = KD(k, \gamma)$, (Shams et al. [15]);

    (iv) $\lim_{q \to 1^-} k - UB_q(\alpha,0,0) = UM(\alpha,k)$, (Kanas [15]);

    (v) $\lim_{q \to 1^-} 0 - UB_q(\alpha,0,0) = M_\alpha$, (Mocanu [13]).

Geometrically, a function $f \in \mathsf{A}$ is said to be in the class $k - UB_q(\alpha, \beta, \gamma)$, if and only if, the function $J_q(\alpha, \beta, \gamma, f(z))$ takes all values in the conic domain Ω_k which is defined as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}$$

This domain represents the right half plane when $k = 0$, a hyperbola when $0 < k < 1$, a parabola when $k = 1$ and an ellipse when $k > 1$. The functions, which play the role of extremal functions of these conic regions are given as follows

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\[2mm] 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\[2mm] 1 + \dfrac{2}{1-k^2}\sinh^2\left[(\dfrac{2}{\pi}\arccos k)\arctan h\sqrt{z} \right], & 0 < k < 1, \\[2mm] \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx \right) + \dfrac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.5}$$

where $u(z) = \dfrac{z - \sqrt{t}}{1 - \sqrt{tz}}$, $t \in (0,1)$ $z \in \mathsf{U}$ and z is chosen such that $k = \cosh\left(\dfrac{\pi R'(t)}{4R(t)}\right)$,

$R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, [11, 12]. Equivalently a function $f \in \mathsf{A}$ is said to be in the class $k - UB_q(\alpha, \beta, \gamma)$, $k \geq 0$, if and only if, $J_q(\alpha, \beta, \gamma, f(z)) \prec p_k(z)$, where $p_k(z)$ is defined by (1.5), $\alpha \geq 0$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$.

By virtue of the properties of the domains for $p \prec p_k$, we have

$$\Re p(z) \geq \Re(p_k(z)) > \frac{k}{k+1}. \tag{1.6}$$

Lemma 1.2. *(Jack's lemma)[7] Let $B(z)$ be analytic in U with $B(0) = 0$. If the maximum value of the $|B(z)|$ on the circle $|z| = r$, $(0 < r < 1)$ is attained at $z = z_0$, then there is a real number λ, $\lambda \geq 1$, such that*

$$\frac{z_0 B'(z_0)}{B(z_0)} = \lambda.$$

Lemma 1.3. *If $f(z) \prec H(z)$ and $g(z) \prec H(z)$, then for $t \in [0,1]$,*
$$(1-t)f(z) + tg(z) \prec H(z).$$

2. Main results

We need the following lemmas, to prove our main results.

Lemma 2.1. *(q -Jack's lemma)[17] Let $B(z)$ be analytic in U with $B(0) = 0$. If the maximum value of the $|B(z)|$ on the circle $|z| = r$, is attained at $z = z_0 \in \mathsf{U}$, then there is a real number λ, $\lambda \geq 1$, such that*

$$\frac{z_0 \partial_q B(z_0)}{B(z_0)} = \lambda.$$

Proof: Using the definition of q -difference operator and Jack's Lemma (1.2), then we have

$$\partial_q B(z) = \frac{B(z) - B(qz)}{(1-q)z} = \frac{B(z) - B(z_0)}{(z - z_0)}, \qquad qz = z_0.$$

If we take the limit for $z \to z_0$ we get

$$\lim_{z \to z_0} \partial_q B(z) = \partial_q B(z_0) = \lim_{z \to z_0} \frac{B(z) - B(z_0)}{(z - z_0)} = B'(z_0).$$

Therefore we have

$$z_0 B'(z_0) = \lambda B(z_0) = z_0 \partial_q B(z_0). \text{ Hence the result.}$$

Lemma 2.2. *[4] Let $f(z) \in \mathsf{A}$, then*

$$\partial_q \{\log(f(z))\} = \frac{\ln q}{(q-1)} \frac{\partial_q f(z)}{f(z)}$$

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Proof: $\quad \partial_q\{\log(f(z))\} = \partial_q \log(f(z))\partial_q f(z),$

but

$$\partial_q \log(f(z)) = \frac{\log(qf(z)) - \log(f(z))}{(q-1)f(z)} = \frac{\ln q}{(q-1)f(z)}.$$

Therefore

$$\partial_q\{\log(f(z))\} = \frac{\ln q}{(q-1)} \frac{\partial_q f(z)}{f(z)}.$$

Lemma 2.3. *Let* $p(z) = 1 + p_n z^n + ...,\quad n \geq 1,$ *be analytic in the unit disc* U *and let* $g(z) = 1 + c_1 z + ...,$ *be analytic and univalent in* $\overline{\mathsf{U}}$. *If* $p(z)$ *is not subordinate to* $g(z)$, *then there exists a real number* λ $(\lambda \geq 1)$, $z_0 \in \mathsf{U}$ *and* $\zeta_0 \in \partial\mathsf{U}$ *such that*

1. $p(|z| < |z_0|) \subset g(\mathsf{U})$.
2. $p(z_0) = g(\zeta_0)$.
3. $\arg(z_0 \partial_q p(z_0)) = \arg(\zeta_0 \partial_q g(\zeta_0))$.
4. $|z_0 \partial_q p(z_0)| = \lambda |\zeta_0 \partial_q g(\zeta_0)|$.

Proof: Suppose, $p(z)$ is not subordinate to $g(z)$, then there is $z_0 \in \mathsf{U}$ such that $p(|z| < |z_0|) \subset g(\mathsf{U})$ and $p(z_0) \in \partial g(\mathsf{U})$. Now let us define $F(z) = g^{-1}(p(z))$, then $F(z)$ satisfies the hypotheses of Lemma 2.1 at $z = z_0$, and hence the result.

Lemma 2.4. *Let* β *and* γ *be complex numbers with* $\beta \neq 0$, *and let* $h(z)$ *be a regular in* U *with* $h(0) = 1$ *and* $\qquad Re[\beta h(z) + \gamma] > 0.$ $\qquad\qquad$ (2.1)

If $p(z) = 1 + p_1 z + ...$ is analytic in U, then

$$p(z) + \frac{z\partial_q p(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \qquad (2.2)$$

Proof: Let us assume that all functions under consideration are analytic in the closed disc $\overline{\mathsf{U}}$. Suppose that $p(z)$ is not subordinate to $g(z)$, then by lemma 2.3, there are $z_0 \in \mathsf{U}$, $\zeta_0 \in \partial\mathsf{U}$ and λ, $\lambda \geq 1$, such that

$$p(z_0) + \frac{z_0\partial_q p(z_0)}{\beta p(z_0) + \gamma} = h(\zeta_0) + \frac{\zeta_0\partial_q h(\zeta_0)}{\beta h(\zeta_0) + \gamma}. \qquad (2.3)$$

From (2.1), we have

$$\arg|\beta h(\zeta_0) + \gamma| < \frac{\pi}{2},$$

and $\zeta_0 \partial_q h(\zeta_0)$ is in the direction of the outer normal to the convex domain $h(\mathsf{U})$, therefore the right-hand of (2.3) is a complex number outside $h(\mathsf{U})$.
Therefore

$$p(z_0) + \frac{z_0 \partial_q p(z_0)}{\beta p(z_0) + \gamma} \notin h(\mathsf{U}),$$

and this contradicts the hypothesis.

Hence $p(z) \prec h(z)$. To remove the restriction, we need to replace $p(z)$ by $p_\tau = p(\tau z)$ and $h_\tau = h(\tau z)$, $0 < \tau < 1$. All the hypotheses of the theorem are satisfied and hence we conclude that $p_\tau(z) \prec h_\tau(z)$ for each τ. By letting $\tau \to 1$, we get $p(z) \prec h(z)$.

Lemma 2.5. *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be regular in U with $h(0) = 1$ and $Re[\beta h(z) + \gamma] > 0$. Let $p(z)$ is analytic in U, $p(0) = 1$ and satisfies*

$$p(z) + \frac{z \partial_q p(z)}{\beta p(z) + \gamma} \prec h(z). \tag{2.4}$$

and $g(z)$ is the analytic solution of

$$g(z) + \frac{z \partial_q g(z)}{\beta g(z) + \gamma} = h(z), \tag{2.5}$$

then $g(z)$ is univalent, $p(z) \prec g(z) \prec h(z)$.

Proof: Let us assume that all functions under consideration are analytic in the closed disc $\overline{\mathsf{U}}$. Then (2.5) and Lemma 2.4 imply $p(z) \prec h(z)$ and $g(z) \prec h(z)$. If $p(z) {}^0 h(z)$, then by lemma 2.3 there are $z_0 \in \mathsf{U}$, $\zeta_0 \in \partial \mathsf{U}$ and λ, $\lambda \geq 1$, such that

$$p(z_0) + \frac{z_0 \partial_q p(z_0)}{\beta p(z_0) + \gamma} = g(\zeta_0) + \frac{\lambda \zeta_0 \partial_q g(\zeta_0)}{\beta g(\zeta_0) + \gamma}.$$

From (2.4) and (2.5), we get

$$p(z_0) + \frac{z_0 \partial_q p(z_0)}{\beta p(z_0) + \gamma} = g(\zeta_0) + \lambda[h(\zeta_0) - g(\zeta_0)] \in h(\mathsf{U}). \tag{2.6}$$

Now $h(\mathsf{U})$ is a convex domain, and $h(\zeta_0) \in \partial h(\mathsf{U})$, $\lambda \geq 1$ and $[h(\zeta_0) - g(\zeta_0)]$ points out of the domain. And this contradicts (2.6). Hence we conclude that $p(z) \prec g(z)$.

To remove the restriction on the functions involved, we introduce $p(\tau z)$, $g(\tau z)$ and $h(\tau z)$. Hence the proof.

Theorem 2.6. *A function $f \in \mathsf{A}$ and of the form (1.1) is in the class $k - UB_q(\alpha, \beta, \gamma)$ if*

$$\sum_{n=2}^{\infty} \Psi_n(k, \alpha, \beta, \gamma, q) < (1 - \beta)(1 - \gamma), \tag{2.7}$$

where

$$\Psi_n(k, \alpha, \beta, \gamma, q) = (k+1)\{([n]_q - 1)(1-\alpha)(1-\gamma) + \alpha(1-\beta)[n]_q([n]_q - 1)\}|a_n|$$

$$+ (k+1)\sum_{j=2}^{n-1}\left\{([j]_q - 1)(1-\alpha)(1-\gamma) + [n-j]_q\alpha(1-\beta)\right\}[n+1-j]_q\left|a_j a_{n+1-j}\right|$$

$$+ (1-\beta)(1-\gamma)([n]_q + 1)\,|\,a_n\,| + (1-\beta)(1-\gamma)\sum_{j=2}^{n-1}[n+1-j]_q\,|\,a_j a_{n+1-j}\,|.$$

where $k \geq 0, \alpha \geq 0, 1 > \beta \geq 0, 1 > \gamma \geq 0, 1 > q > 0$

Proof: Assuming that (2.7) holds, then it suffices to show that

$$k\,|\,J_q(\alpha,\beta,\gamma,f(z))-1\,| - Re\{J_q(\alpha,\beta,\gamma,f(z))-1\} < 1.$$

Now, consider

$$|\,J_q(\alpha,\beta,\gamma,f(z))-1\,| = \left|\frac{1-\alpha}{1-\beta}\left\{\frac{z\partial_q f(z)}{f(z)} - \beta\right\} + \frac{\alpha}{1-\gamma}\left\{1-\gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\} - 1\right|$$

$$= \left|\frac{1-\alpha}{1-\beta}\frac{zf(z)\partial_q f(z)}{f(z)} - \frac{(1-\alpha)\beta}{1-\beta} + \alpha + \frac{\alpha}{1-\gamma}\frac{qz\partial_q^2 f(z)}{\partial_q f(z)} - 1\right|$$

$$= \left|\frac{1-\alpha}{1-\beta}\frac{zf(z)\partial_q f(z)}{f(z)} - \frac{1-\alpha}{1-\beta} + \frac{\alpha}{1-\gamma}\frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right|$$

$$= \left|\frac{(1-\alpha)(1-\gamma)z\partial_q f(z)\partial_q f(z) - (1-\alpha)(1-\gamma)f(z)\partial_q f(z) + \alpha(1-\beta)qzf(z)\partial_q^2 f(z)}{(1-\beta)(1-\gamma)f(z)\partial_q f(z)}\right|.$$

$$(2.8)$$

Now, from (1.1) and (1.2), we have

$$z\partial_q f(z)\partial_q f(z) = z\left(\sum_{n=0}^{\infty}[n]_q a_n z^{n-1}\right)\left(\sum_{n=0}^{\infty}[n]_q a_n z^{n-1}\right), a_0 = 0, a_1 = 1, [0]_q = [1]_q = 1$$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty}[n]_q a_n z^n\right)\left(\sum_{n=0}^{\infty}[n]_q a_n z^n\right) \quad = \frac{1}{z}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}[j]_q[n-j]_q a_j a_{n-j}\right)z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}[j]_q[n-j]_q a_j a_{n-j}\right)z^{n-1} = z + \sum_{n=3}^{\infty}\left(\sum_{j=0}^{n}[j]_q[n-j]_q a_j a_{n-j}\right)z^{n-1}$$

$$= z + \sum_{n=2}^{\infty}\left(\sum_{j=0}^{n+1}[j]_q[n+1-j]_q a_j a_{n+1-j}\right)z^n$$

$$= z + \sum_{n=2}^{\infty}\left(2[n]_q a_n + \sum_{j=2}^{n-1}[j]_q[n+1-j]_q a_j a_{n+1-j}\right)z^n.$$

Similarly, we can get

$$f(z)\partial_q f(z) = z + \sum_{n=2}^{\infty}\left(([n]_q + 1)a_n + \sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right)z^n$$

And

$$f(z)\partial_q^2 f(z) = \sum_{n=2}^{\infty}\left([n]_q([n]_q-1)a_n + \sum_{j=2}^{n-1}[n+1-j]_q[n-j]_q a_j a_{n+1-j} \right)z^n.$$

Using the equalities (2.8), we get

$$|J_q(\alpha,\beta,\gamma,f(z))-1| = \frac{\sum_{n=2}^{\infty}\left[([n]_q-1)(1-\alpha)(1-\gamma)+q\alpha(1-\beta)[n]_q([n]_q-1)\right]a_n z^n}{(1-\beta)(1-\gamma)\left[z+\sum_{n=2}^{\infty}\left[([n]_q+1)a_n+\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right]z^n\right]}$$

$$+\frac{\sum_{n=2}^{\infty}\left[\sum_{j=2}^{n-1}([j]_q-1)[n+1-j]_q(1-\alpha)(1-\gamma)a_j a_{n+1-j}\right]z^n}{(1-\beta)(1-\gamma)\left[z+\sum_{n=2}^{\infty}\left[([n]_q+1)a_n+\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right]z^n\right]}$$

$$+\frac{\sum_{n=2}^{\infty}\left[\sum_{j=2}^{n-1}[n+1-j]_q[n-j]_q q\alpha(1-\beta)a_j a_{n+1-j}\right]z^n}{(1-\beta)(1-\gamma)\left[z+\sum_{n=2}^{\infty}\left[([n]_q+1)a_n+\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right]z^n\right]}$$

$$\leq \frac{\sum_{n=2}^{\infty}\left|([n]_q-1)(1-\alpha)(1-\gamma)+q\alpha(1-\beta)[n]_q([n]_q-1)\right||a_n|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}$$

$$+\frac{\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}([j]_q-1)[n+1-j]_q(1-\alpha)(1-\gamma)a_j a_{n+1-j}\right|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}$$

$$+\frac{\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q[n-j]_q q\alpha(1-\beta)a_j a_{n+1-j}\right|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}.$$

Since

$$k|J_q(\alpha,\beta,\gamma,f(z))-1|-\Re\{J_q(\alpha,\beta,\gamma,f(z))-1\}$$
$$\leq (k+1)|J_q(\alpha,\beta,\gamma,f(z))-1|$$

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$$\leq \frac{(k+1)\sum_{n=2}^{\infty}\left|([n]_q-1)(1-\alpha)(1-\gamma)+q\alpha(1-\beta)[n]_q([n]_q-1)\right||a_n|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}$$

$$+\frac{(k+1)\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}([j]_q-1)[n+1-j]_q(1-\alpha)(1-\gamma)a_j a_{n+1-j}\right|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}$$

$$+\frac{(k+1)\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q[n-j]_q q\alpha(1-\beta)a_j a_{n+1-j}\right|}{(1-\beta)(1-\gamma)\left[1-\sum_{n=2}^{\infty}([n]_q+1)|a_n|-\sum_{n=2}^{\infty}\left|\sum_{j=2}^{n-1}[n+1-j]_q a_j a_{n+1-j}\right|\right]}.$$

The last expression is bounded by 1 if

$$\sum_{n=2}^{\infty}(k+1)\left|([n]_q-1)(1-\alpha)(1-\gamma)+q\alpha(1-\beta)[n]_q([n]_q-1)\right||a_n|$$

$$+\sum_{n=2}^{\infty}\left\{\sum_{j=2}^{n-1}(k+1)\left|([j]_q-1)[n+1-j]_q(1-\alpha)(1-\gamma)a_j a_{n+1-j}\right|\right\}$$

$$+\sum_{n=2}^{\infty}\left\{\sum_{j=2}^{n-1}(k+1)\left|[n+1-j]_q[n-j]_q q\alpha(1-\beta)a_j a_{n+1-j}\right|\right\}$$

$$+\sum_{n=2}^{\infty}(1-\beta)(1-\gamma)([n]_q+1)|a_n|+\sum_{n=2}^{\infty}(1-\beta)(1-\gamma)\sum_{j=2}^{n-1}[n+1-j]_q |)a_j a_{n+1-j}|$$

$$< (1-\beta)(1-\gamma).$$

This complete the proof.

As $q \to 1$ in the above theorem we get the flowing result proved by Noor and Malik in [14], which reads.

Corollary 2.7. *A function f of the form (1.1) is in the class $k-UB(\alpha,\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty}\Psi_n(k,\alpha,\beta,\gamma) < (1-\beta)(1-\gamma)$$

where

$$\Psi_n(k,\alpha,\beta,\gamma) = (k+1)\{(n-1)(1-\alpha)(1-\gamma)+n\alpha(1-\beta)(n-1)\}|a_n|$$

$$+(k+1)\sum_{j=2}^{n-1}\{(j-1)(1-\alpha)(1-\gamma)+(n-j)\alpha(1-\beta)\}(n+1-j)\left|a_j a_{n+1-j}\right| \quad (2.9)$$

$$+(1-\beta)(1-\gamma)(n+1)|a_n|+(1-\beta)(1-\gamma)\sum_{j=2}^{n-1}(n+1-j)|a_j a_{n+1-j}|$$

where $k \geq 0, \alpha \geq 0, 1 > \beta \geq 0, 1 > \gamma \geq 0$.

As $q \to 1$ and for $\alpha = 0$, we get the following result, proved by Shams and Kulkarni [15].

Corollary 2.8. *A function* $f \in \mathbb{A}$ *and of the form (1.1) in the class* $SD(k, \beta)$, *if it satisfies the condition*

$(1 - \beta)(1 - \gamma) >$

$$\sum_{n=2}^{\infty} \left\{ (k+1)(n-1)(1-\gamma) \mid a_n \mid + (k+1) \sum_{j=2}^{n-1} (j-1)(1-\gamma)(n+1-j) \left| a_j a_{n+1-j} \right| \right\}$$

$$+ \sum_{n=2}^{\infty} \left\{ (1-\beta)(1-\gamma) \mid a_n \mid + (1-\beta)(1-\gamma) \sum_{j=2}^{n-1} (n+1-j) \mid a_j a_{n+1-j} \mid \right\}$$

$$> (1-\gamma) \sum_{n=2}^{\infty} \left\{ (k+1)(n-1) + (1-\beta) \right\} \mid a_n \mid.$$

$$\tag{2.10}$$

This implies that

$$\sum_{n=2}^{\infty} \left\{ n(k+1) - (k+\beta) \right\} \mid a_n \mid < 1 - \beta.$$

As $q \to 1$ and for $\alpha = 1$, we have the following result, proved by Shams and Kulkarni [15].

Corollary 2.9. *A function* $f \in \mathbb{A}$ *and of the form (1.1) in the class* $KD(k, \gamma)$, *if it satisfies the condition*

$(1 - \beta)(1 - \gamma)$

$$> \sum_{n=2}^{\infty} \left\{ n(k+1)(n-1)(1-\beta) \mid a_n \mid + (k+1) \sum_{j=2}^{n-1} (1-\beta)(n-j)(n+1-j) \left| a_j a_{n+1-j} \right| \right\}$$

$$+ \sum_{n=2}^{\infty} \left\{ n(1-\beta)(1-\gamma) \mid a_n \mid + (1-\beta)(1-\gamma) \sum_{j=2}^{n-1} (n+1-j) \mid a_j a_{n+1-j} \mid \right\}$$

$$> (1-\beta) \sum_{n=2}^{\infty} n \left\{ (k+1)(n-1) + (1-\gamma) \right\} \mid a_n \mid.$$

This implies that

$$\sum_{n=2}^{\infty} n \left\{ n(k+1) - (k+\gamma) \right\} \mid a_n \mid < 1 - \gamma.$$

When $\beta = 0$, $\gamma = 0$ and as $q \to 1$, we get the well-known Kanas's result [10].

Corollary 2.10. *A function* $f \in \mathbb{A}$ *and of the form (1.1) in the class* $UM(\alpha, k)$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty} \Psi_n(k,\alpha) < 1,$$

$$\Psi_n(k,\alpha) = (k+1)(n-1)(1-\alpha+n\alpha)\,|\,a_n\,|$$

$$+(k+1)\sum_{j=2}^{n-1}\{(j-1)(1-\alpha)+\alpha(n-j)\}|\,a_j a_{n+1-j}\,| + (n+1)\,|\,a_n\,|$$

where

$$+\sum_{j=2}^{n-1}(n+1-j)\,|\,a_j a_{n+1-j}\,|.$$

For $\alpha=0$, $\beta=0$ and as $q\to 1$, we get the below result, proved by Kanas and Wisniowska [12].

Corollary 2.11. *A function $f\in \mathsf{A}$ and of the form (1.1) in the class $k-ST$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n+(k(n-1)\}|\,a_n\,| < 1.$$

As $q\to 1$, $\alpha=0$, $k=0$, then we get result, proved by Silverman in[16].

Corollary 2.12. *A function $f\in \mathsf{A}$ and of the form (1.1) in the class $S^*(\beta)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}(n-\beta)|\,a_n\,| < 1-\beta.$$

Theorem 2.13. *If $f(z)\in k-UB_q(\alpha,\beta,\gamma)$. Then $f(z)\in SD_q(k,\zeta)$, where*

$$\zeta = \frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}. \tag{2.12}$$

Proof: Let $\dfrac{1}{1-\zeta}\left\{\dfrac{z\partial_q f(z)}{f(z)}-\zeta\right\}=p(z),$

where $p(z)$ is analytic in U and $p(0)=1$. Therefore

$$z\partial_q f(z) = \left[(1-\zeta)p(z)+\zeta\right]f(z). \tag{2.13}$$

Using q-logarithmic differentiation of (2.13), we get

$$\frac{\partial_q(z\partial_q f(z))}{z\partial_q f(z)} = \frac{(1-\zeta)\partial_q p(z)}{(1-\zeta)p(z)+\zeta} + \frac{\partial_q f(z)}{f(z)},$$

which implies that

$$1+\frac{qz\partial_q^2 f(z)}{\partial_q f(z)} = \frac{(1-\zeta)z\partial_q p(z)}{\left[(1-\zeta)p(z)+\zeta\right]} + (1-\zeta)p(z)+\zeta. \tag{2.14}$$

Using (2.13) and (2.14) in (1.4), we get

$$J_q(\alpha,\beta,\gamma,f(z)) = \frac{1-\alpha}{1-\beta}\{(1-\zeta)p(z)+\zeta-\beta\}$$

$$+\frac{\alpha}{1-\gamma}\left\{(1-\zeta)p(z)+\zeta+\frac{(1-\zeta)z\partial_q p(z)}{[(1-\zeta)p(z)+\zeta]}-\gamma\right\}$$

$$=\frac{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}{(1-\gamma)(1-\beta)}\{(1-\zeta)p(z)+\zeta\}-\frac{\beta(1-\gamma)(1-\alpha+\alpha\gamma(1-\beta))}{(1-\gamma)(1-\beta)}$$

$$+\frac{\alpha}{1-\gamma}\frac{(1-\zeta)z\partial_q p(z)}{[(1-\zeta)p(z)+\zeta]}.$$

Now for $\zeta = \dfrac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}$, we have

$$J_q(\alpha,\beta,\gamma,f(z)) = p(z)+\frac{z\partial_q p(z)}{\dfrac{1-\gamma}{\alpha}p(z)+\dfrac{\zeta}{1-\zeta}}.$$

Since $f(z)\in k-UB_q(\alpha,\beta,\gamma)$, we get

$$p(z)+\frac{z\partial_q p(z)}{\dfrac{1-\gamma}{\alpha}p(z)+\dfrac{\zeta}{1-\zeta}}\prec p_k(z),$$

where $p_k(z)$ is defined by (1.5). Since $Re\left\{\dfrac{1-\gamma}{\delta}\dfrac{k}{k+1}+\dfrac{\zeta}{1-\zeta}\right\}>0,\ z\in \mathsf{U}$, applying Lemma 2.5, we get

$$\frac{1}{1-\zeta}\left\{\frac{z\partial_q f(z)}{f(z)}-\zeta\right\}=p(z)\prec p_k(z),\ \text{which implies that}\ f(z)\in SD_q(k,\zeta).$$

Theorem 2.14. *Let $f\in \mathsf{A}$. Then $f(z)\in k-UB_q(\alpha,\beta,\gamma),\ \alpha\neq 0$,if and only if, there is a function $g(z)\in k-ST_q$ such that*

$$f(z)=\left[m\int_0^z t^{m-1}(\frac{g(t)}{t})^{\frac{1-\gamma}{\alpha}}d_q t\right]^{\frac{1}{m}}=z+...,\qquad (2.15)$$

where $m=1+\dfrac{(1-\alpha)(1-\gamma)}{\alpha(1-\beta)}$.

Proof: From (2.15), we have

$$\left(z\partial_q f(z)\right)\left(f(z)\right)^{m-1}=z^{\{}m-\frac{1-\gamma}{\alpha}-1\}\left(g(z)\right)^{\frac{1-\gamma}{\alpha}}.\qquad (2.16)$$

Using q-logarithmic differentiation of (2.16), we get

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$$\frac{\partial_q(z\partial_q f(z))}{z\partial_q f(z)} + (m-1)\frac{\partial_q f(z)}{f(z)} = \{m - \frac{1-\gamma}{\alpha} - 1\}\frac{1}{z} + (\frac{1-\gamma}{\alpha})\frac{\partial_q g(z)}{g(z)}.$$

This implies that

$$1 + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)} + (m-1)\frac{z\partial_q f(z)}{f(z)} = (m - \frac{1-\gamma}{\alpha}) + (\frac{1-\gamma}{\alpha})\frac{z\partial_q g(z)}{g(z)}.$$

Substituting for $m = 1 + \frac{(1-\alpha)(1-\gamma)}{\alpha(1-\beta)}$, we get

$$\frac{1-\alpha}{1-\beta}\left\{\frac{z\partial_q f(z)}{f(z)} - \beta\right\} + \frac{\alpha}{1-\gamma}\left\{1 - \gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\} = \frac{z\partial_q g(z)}{g(z)}. \tag{2.17}$$

Now from the definition, we get the required result.

Theorem 2.15. *Let* $f(z) \in k - UB_q(\alpha, \beta, \gamma)$. *Then the function*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1-\alpha}{1-\beta}}\left(\partial_q f(z)\right)^{\frac{\alpha}{1-\gamma}} \tag{2.18}$$

belongs to $k - ST_q$ *for all* $z \in \mathsf{U}$.

Proof: Using q-logarithmic differentiation of (2.18), we get

$$\frac{\partial_q g(z)}{g(z)} = \frac{1}{z} + \frac{1-\alpha}{1-\beta}\frac{z\partial_q f(z)}{f(z)} - \frac{1-\alpha}{1-\beta}\frac{1}{z} - \frac{\alpha}{1-\gamma}\frac{1}{z} + \frac{\alpha}{1-\gamma}\frac{qz\partial_q^2 f(z)}{\partial_q f(z)}.$$

This implies that

$$\frac{z\partial_q g(z)}{g(z)} = \frac{1-\alpha}{1-\beta}\frac{z\partial_q f(z)}{f(z)} + \frac{\alpha}{1-\gamma}\left\{1 - \gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\},$$

and the result follows since $f \in k - UB_q(\alpha, \beta, \gamma)$.

Theorem 2.16. *For* $\alpha > \alpha_1 \geq 0$, $k - UB_q(\alpha, \beta, \gamma) \subset k - UB_q(\alpha_1, \beta, \gamma)$.

Proof: Let $f(z) \in k - UB_q(\alpha, \beta, \gamma)$. Then

$$J_q(\alpha_1, \beta, \gamma, f(z))$$

$$= \frac{1-\alpha_1}{1-\beta}\left\{\frac{z\partial_q f(z)}{f(z)} - \beta\right\} + \frac{\alpha_1}{1-\gamma}\left\{1 - \gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\}$$

$$= \left(1 - \frac{\alpha_1}{\alpha}\right)J_q(0, \beta, \gamma, f(z)) + \frac{\alpha_1}{\alpha}J_q(\alpha, \beta, \gamma, f(z))$$

$$= \left(1 - \frac{\alpha_1}{\alpha}\right)\frac{1}{1-\beta}\left\{\frac{z\partial_q f(z)}{f(z)} - \beta\right\} + \frac{\alpha_1}{\alpha}\left[\frac{1-\alpha_1}{1-\beta}\left\{\frac{z\partial_q f(z)}{f(z)} - \beta\right\} + \frac{\alpha}{1-\gamma}\left\{1 - \gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)}\right\}\right]$$

$$= \left(1 - \frac{\alpha_1}{\alpha}\right) p_1(z) + \frac{\alpha_1}{\alpha} p_2(z)$$

$$= p(z),$$

where $p_1(z) = \dfrac{1}{1-\beta} \left\{ \dfrac{z\partial_q f(z)}{f(z)} - \beta \right\} \in P(q)$

and

$$p_2(z) = \frac{\alpha_1}{\alpha} \left[\frac{1-\alpha_1}{1-\beta} \left\{ \frac{z\partial_q f(z)}{f(z)} - \beta \right\} + \frac{\alpha}{1-\gamma} \left\{ 1 - \gamma + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)} \right\} \right] \in P(q).$$

Now $f(z) \in k - UB_q(\alpha, \beta, \gamma)$, therefore

$$J_q(\alpha, \beta, \gamma, f(z)) = p(z) \prec p_k(z).$$

Using Theorem 2.13, we obtain

$$J_q(0, \beta, \gamma, f(z)) \prec p_k(z).$$

Using these along with lemma 1.3, we get

$$J_q(\alpha_1, \beta, \gamma, f(z)) \prec p_k(z),$$

which implies that $f(z) \in k - UB_q(\alpha_1, \beta, \gamma)$.

The q-Bernardi integral operator is defined as the following: Let $f(z) \in \mathsf{A}$. Then $L_B : \mathsf{A} \to \mathsf{A}$ is said to be q-Bernardi operator $L_B(f) = F_a$, and

$$F_a(z) = \frac{[a+1]_q}{z^a} \int_0^z t^{a-1} f(t) d_q t = \sum_{n=1}^{\infty} \left(\frac{[a+1]_q}{[a+n]_q} \right) a_n z^n. \tag{2.19}$$

Now we prove the following.

Theorem 2.17. *Let* $f(z) \in k - UB_q(\alpha, \beta, \gamma)$. *Then* $F_a(z) \in SD_q(k, \zeta)$, *where* ζ *is defined by (2.12).*
Proof: From (2.19), we have

$$z^a F_a(z) = [a+1]_q \int_0^z t^{a-1} f(t) d_q t.$$

which implies that

$$[a+1]_q f(z) = [a]_q F_a(z) + z\partial_q F_a(z). \tag{2.20}$$

Using q-logarithmic differentiation of (2.20), we get

$$\frac{z\partial_q f(z)}{f(z)} = \frac{z\partial_q \big(z\partial_q F_a(z) \big) + [a]_q z\partial_q F_a(qz)}{z\partial_q F_a(z) + [a]_q F_a(qz)}$$

$$= \frac{\dfrac{z\partial_q \big(z\partial_q F_a(z) \big)}{F_a(qz)} + [a]_q \dfrac{z\partial_q F_a(qz)}{F_a(qz)}}{\dfrac{z\partial_q F_a(z)}{F_a(qz)} + [a]_q}. \tag{2.21}$$

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Let $\dfrac{z\partial_q F_a(qz)}{F_a(qz)} = (1-\zeta)p(z) + \zeta$ with $p(z)$ is analytic in U and $p(z) = 1$. Then

$$z\partial_q F_a(z) = [(1-\zeta)p(z) + \zeta]F_a(qz).$$

Using q-differentiation, we get

$$\partial_q\big(z\partial_q F_a(z)\big) = [(1-\zeta)p(z) + \zeta]F_a(qz) + [(1-\zeta)p(z)]F_a(q^2 z)$$

which implies

$$\frac{z\partial_q\big(z\partial_q F_a(z)\big)}{F_a(qz)} = [(1-\zeta)p(z) + \zeta]^2 + [(1-\zeta)z\partial_q p(z)]\frac{F_a(q^2 z)}{F_a(qz)}.$$

But $\dfrac{F_a(q^2 z)}{F_a(qz)} = q^a$ by (2.19), therefore

$$\frac{z\partial_q\big(z\partial_q F_a(z)\big)}{F_a(qz)} = [(1-\zeta)p(z) + \zeta]^2 + [(1-\zeta)q^a z\partial_q p(z)] \tag{2.22}$$

Using (2.22) in (2.21), we get

$$
\begin{aligned}
\frac{z\partial_q f(z)}{f(z)} &= \frac{[(1-\zeta)p(z) + \zeta]^2 + [(1-\zeta)q^a z\partial_q p(z)] + [a]_q[(1-\zeta)p(z) + \zeta]}{(1-\zeta)p(z) + \zeta + [a]_q} \\
&= (1-\zeta)p(z) + \zeta + \frac{(1-\zeta)q^a z\partial_q p(z)}{(1-\zeta)p(z) + \zeta + [a]_q} \\
&= (1-\zeta)h(z) + \zeta. \tag{2.23}
\end{aligned}
$$

where

$$h(z) = p(z) + \frac{q^a z\partial_q p(z)}{(1-\zeta)p(z) + \zeta + [a]_q}. \tag{2.24}$$

Using q-logarithmic differentiation of (2.23), we get

$$1 + \frac{qz\partial_q^2 f(z)}{\partial_q f(z)} = (1-\zeta)h(z) + \zeta + \frac{(1-\zeta)z\partial_q h(z)}{(1-\zeta)h(z) + \zeta}. \tag{2.25}$$

Using (2.23) and (2.25), we get

$$
\begin{aligned}
J_q(\alpha, \beta, \gamma, f(z)) &= \frac{1-\alpha}{1-\beta}\big\{(1-\zeta)h(z) + \zeta - \beta\big\} \\
&\quad + \frac{\alpha}{1-\gamma}\left\{(1-\zeta)h(z) + \zeta + \frac{(1-\zeta)z\partial_q h(z)}{(1-\zeta)h(z) + \zeta} - \gamma\right\} \\[4pt]
. \\
&= \frac{(1-\alpha)(1-\gamma) + \alpha(1-\beta)}{(1-\alpha)(1-\gamma)}[(1-\zeta)h(z) + \zeta] - \frac{\beta(1-\alpha)(1-\gamma) + \alpha\gamma(1-\beta)}{(1-\alpha)(1-\gamma)} \\
&\quad + \frac{\alpha}{1-\gamma}\frac{(1-\zeta)z\partial_q h(z)}{(1-\zeta)h(z) + \zeta}.
\end{aligned}
$$

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Now for $\zeta = 1 - \dfrac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma) + \alpha(1-\beta)}$, we have

$$J_q(\alpha, \beta, \gamma, f(z)) = \frac{z\partial_q h(z)}{\dfrac{1-\gamma}{\alpha} h(z) + \dfrac{\zeta}{1-\zeta}}.$$

Since $f(z) \in k - UB_q(\alpha, \beta, \gamma)$, we have

$$h(z) + \frac{z\partial_q h(z)}{\dfrac{1-\gamma}{\alpha} h(z) + \dfrac{\zeta}{1-\zeta}} \prec p_k(z),$$

where $p_k(z$ is defined by (1.5). Since $\Re\{\dfrac{1-\gamma}{\alpha}\dfrac{k}{k+1} + \dfrac{\zeta}{1-\zeta}\} > 0, \quad z \in \mathsf{U},$ applying Lemma 2.5, we get

$$p(z) + \frac{q^a z \partial_q p(z)}{(1-\zeta)p(z) + \zeta + [a]_q} = h(z) \prec p_k(z).$$

Now as $\Re\{(1-\zeta)\dfrac{k}{k+1} + \zeta + [a]_q\} > 0, \quad z \in \mathsf{U},$ again applying Lemma 2.5, we get $p(z) \prec p_k(z)$, which implies that $F_a(z) \in SD_q(k, \zeta)$. This completes the proof.

As $q \to 1$ in the above theorem we get the flowing result proved by Noor and Malik in [14].

Corollary 2.18. *Let* $f(z) \in k - UB(0, \beta, \gamma)$. *Then* $F_a(z) \in SD(k, \zeta)$, *where* ζ *is defined by (2.12).*
As $q \to 1$ and for $\alpha = 0$, we get the following result, proved by Shams and Kulkarni [15].

Corollary 2.19. *Let* $f(z) \in k - UB(0, \beta, \gamma)$. *Then* $F_a(z) \in SD(k, \beta)$.
As $q \to 1$ and for $\alpha = \beta = 0$, we get the following result, proved by Bernardi [1].

Corollary 2.20. *Let* $f(z) \in k - UB(0, 0, \gamma)$. *Then* $F_a(z) \in k - ST$.
As $q \to 1$ and for $k = \alpha = \beta = 0$, we get the following result, proved by Bernardi [1].

Corollary 2.21. *Let* $f(z) \in 0 - UB(0, o, \gamma)$. *Then* $F_a(z) \in S^*$.

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