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A Note on the Lattice of *L*-Closure Operators

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Abstract. In this paper, we investigate the lattice structure of the set of all L-closure operators on a given nonempty set X when membership lattice L is a bounded chain. It is proved that in this case, the lattice of all L-closure operators is distributive, modular but not atomic and not complemented. The authors disprove certain known theorems on the above lattices and the correct results are provided.

Keyword: L-topology, Lattice, Chain, L-closure operator.

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1. Introduction

In 1965, the concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh in his classical paper [12]. In 1968, Chang [2] applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Closure spaces which is a generalization of topological spaces were introduced by Cech [1] and then studied by many authors. They have extended many topological concepts to closure operators. Fuzzy closure spaces were first studied by Mashhour and Ghanim [7]. Fuzzy closure spaces are generalization of fuzzy topological spaces. The definition of Mashhour and Ghanim is analogue of $\breve{C}$ech closure spaces. Srivastava et al., [9] have introduced fuzzy closure spaces as analogue of Brikhoff closure spaces. Srivastava and Srivastava [10] have studied the subspace of a fuzzy closure space and introduced the notion of a T_1-fuzzy closure space. Johnson [5] has studied the lattice structure of the set $L(X)$ of all $\breve{C}$ech fuzzy closure operators on a fixed set X and proved that $L(X)$ is a complete lattice but not complemented. Zhou [11] has introduced the concepts of L-closure spaces and the convergence in L-closure spaces.

In this paper, we have investigated the lattice structure of the lattice $LC(X)$ of all L-closure operators on a given non-empty set X when membership lattice L is a bounded chain. In addition, we have identified the infra L-closure operators and their number and established a relation between ultra L-topology and ultra L-closure operator.

2. Preliminaries

Throughout this paper, X stands for a non-empty set, L for a bounded chain with the least element 0 and the greatest element 1, which is a completely distributive lattice with an order reversing involution ' $'$ ' $\left(i.e.\ \forall a,b \in L, a \le b \Rightarrow a' \ge b'\right.$ and for every $\left. a \in L, a'' = a \right)$ and

$L^X = \{ f : f : X \to L$ is a mapping $\}$. The constant function in L^X, taking value α is denoted by $\underline{\alpha}$ and x_γ, where $\gamma(\ne 0) \in L$, denotes the L - fuzzy point defined by

$$x_\gamma(y) = \begin{cases} \gamma & if \quad y = x \\ 0 & otherwise \end{cases}.$$

Any $f \in L^X$ is called as an L-subset of X and the complement of f, denoted by f' is defined by the formula $f'(x) = [f(x)]'$. The following are some important definition reported in [3,6] :

Definition 2.1. *An element of L is called an atom if it is a minimal element of $L \setminus \{0\}$.*

Definition 2.2. *An element of L is called a dual atom if it is a maximal element of $L \setminus \{1\}$.*

Definition 2.3. *Let δ be a nonempty subset of L^X. We call δ an L-topology on X, if δ satisfies the following conditions :*

(1) $\underline{0}, \underline{1} \in \delta$.

(2) if $f, g \in \delta$, then $f \wedge g \in \delta$.

(3) if $\delta_1 \subseteq \delta$, then $\bigvee_{f \in \delta_1} f \in \delta$.

The pair (L^X, δ) is called an L-topological space.

In this paper, we take the definition of L-closure operator as a generalization of fuzzy closure operator in [7].

Definition 2.4. *A $\breve{C}ech$ L-closure operator on a set X is a function $c : L^X \to L^X$ satisfying the following three axioms :*

(1) $c(\underline{0}) = \underline{0}$.

(2) $f \le c(f)$ for every f in L^X.

(3) $c(f \vee g) = c(f) \vee c(g)$ for all $f, g \in L^X$.

For convenience, we call it a L-closure operator on X. Also (X, c) is called L-closure space.

Definition 2.5. *In an L-closure space (X, c), an L-subset f of X is said to be L-closed if $c(f) = f$. An L-subset f of X is L-open if its complement is closed in (X, c).*

The set of all open L-subsets of (X,c) forms an L-topology on X, called the L-topology associated with the L-closure operator c.
Let F be an L-topology on a set X. Then a function $c: L^X \to L^X$ defined by $c(f)=\bar{f}$ for all $f \in L^X$, where $\bar{f}$ is the closure of f in (L^X, F), is an L-closure operator on X called the L-closure operator associated with the L-topology F.
An L-closure operator on a set X is called L-topological if it is the L-closure operator associated with an L-topology on X.

Remark 2.6. *Note that the different L-closure operators can have the same associated L-topology. But different L-topologies can not have the same associated L-closure operator.*

Example 2.7. *Let $X=\{x,y,z\}$ and $L=\{0,\alpha,\beta,1\}$ be a chain with $0<\alpha<\beta<1$. Then the functions c_1, $c_2 : L^X \to L^X$ defined by :*

$$c_1(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ g & if \quad f=x_\alpha \\ \underline{1} & otherwise \end{cases},$$

where $g \in L^X$ is defined as $g(x)=g(y)=1$ and $g(z)=\beta$

$$\text{and } c_2(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ \underline{1} & otherwise \end{cases}$$

are L-closure operators. Associated L-topologies of c_1 and c_2 are same, which is the indiscrete L-topology.

3. Lattice of L-closure operators

Definition 3.1. *Let c_1 and c_2 be L-closure operators on X. Then $c_1 \le c_2$ if and only if $c_2(f) \le c_1(f), \forall f \in L^X$.*

Remark 3.2. *The set $LC(X)$ of all L-closure operators forms a lattice with this relation $\le$. If $c_1, c_2 \in LC(x)$, then the join $c_1 \vee c_2$ and the meet $c_1 \wedge c_2$ are defined respectively by the following formulas :*
$(c_1 \vee c_2)(x) = \min \{c_1(x), c_2(x)\}$ and
$(c_1 \wedge c_2)(x) = \max \{c_1(x), c_2(x)\}$.

Definition 3.3. *The L-closure operator D defined on X by $D(f)=f$ for all $f \in L^X$, is called the discrete L-closure operator.*
The L-closure operator I on X defined by

$$I(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ \underline{1} & otherwise \end{cases},$$

is called the indiscrete L -closure operator.

Remark 3.4. *D and I are the L -closure operators associated with the discrete and indiscrete L -topologies on X respectively. Moreover D is the unique L -closure operator whose associated L -topology is discrete. Also I and D are the smallest and the largest elements of the lattice $LC(X)$ respectively.*

Theorem 3.5. [5] *$LC(X)$ is a complete lattice.*

In [8], we find the following theorem:

Theorem 3.6. [8] *$LC(X)$ is not modular.*

But this result is not true as shown by the following theorem:

Theorem 3.7. *$LC(X)$ is a distributive lattice.*

Proof: Let c_1, c_2 and c_3 be any three elements of $LC(X)$.
Then by definition of $\leq$, we have
$c_1 \vee (c_2 \wedge c_3) =$ min $[c_1,$ max $\{c_2, c_3\}]$
and $(c_1 \vee c_2) \wedge (c_1 \vee c_3) =$ max [min $\{c_1, c_2\},$ min $\{c_1, c_3\}]$.
For any $f \in L^X$ and $x \in X$, assume that $c_1(f) = g_1$, $c_2(f) = g_2$, $c_3(f) = g_3$ and $g_1(x) = \alpha$, $g_2(x) = \beta$, $g_3(x) = \gamma$.
Since $\alpha, \beta, \gamma \in L$ and L is a chain, the following six case arise:
(1) $\gamma < \beta < \alpha$ (2) $\beta < \gamma < \alpha$ (3) $\alpha < \gamma < \beta$ (4) $\gamma < \alpha < \beta$ (5) $\beta < \alpha < \gamma$ (6) $\alpha < \beta < \gamma$.

<u>Case 1</u> : $\gamma < \beta < \alpha$.
Then $\{c_1 \vee (c_2 \wedge c_3)\}(f)(x)$
$=$ min $\{g_1(x),$ max $\{g_2(x), g_3(x)\}\}$
$=$ min $\{\alpha,$ max $\{\beta, \gamma\}\}$
$=$ min $\{\alpha, \beta\} = \beta$
and $\{(c_1 \vee c_2) \wedge (c_1 \vee c_3)\}(f)(x)$
$=$ max{ min $\{g_1(x), g_2(x)\},$ min $\{g_1(x), g_3(x)\}\}$.
$=$ max { min $\{\alpha, \beta\},$ min $\{\alpha, \gamma\}\}$.
$=$ max $\{\beta, \gamma\} = \beta$.
In the same way, it can be checked that the equality
$\{c_1 \vee (c_2 \wedge c_3)\}(f)(x) = \{(c_1 \vee c_2) \wedge (c_1 \vee c_3)\}(f)(x)$ holds good in the remaining five cases also.
Since $f \in L^X$ and $x \in X$ were arbitrary,

$\Rightarrow c_1 \vee (c_2 \wedge c_3) = (c_1 \vee c_2) \wedge (c_1 \vee c_3)$.

$\Rightarrow LC(X)$ is distributive lattice.

Corollary 1. *$LC(X)$ is a modular lattice.*

4. Infra L-closure operators

Definition 4.1. *An L-closure operator on X is called an infra L-closure operator if the only L-closure operator on X strictly smaller than it is I.*

Theorem 4.2. [5] *If $L = [0,1]$, then there is no infra L-closure operator in $LC(X)$.*

In [8], we find the following result :

Let X be any set and $a, b \in X$ such that $a \neq b$. Define $\psi_{a,b} : L^X \to L^X$ by

$$\psi_{a,b}\,(f) = \begin{cases} f & if \quad f = \underline{0} \\ g_{\alpha,b} & if \quad f = a_\alpha \\ \underline{1} & \quad otherwise \end{cases},$$

where α is a dual atom in L and $g_{\alpha,b}$ is defined by

$$g_{\alpha,b}(a) = \begin{cases} 1 & if \quad a \neq b \\ \alpha & if \quad a = b \end{cases}.$$

Theorem 4.3. [8] *An L-closure operator is an infra L-closure operator if and only if it is of the form $\psi_{a,b}$ for some $a, b \in X, a \neq b$.*

But this result is not true because $\psi_{a,b}$ is not even an L-closure operator as shown below:

Let $a_\alpha, a_\eta \in L^X$, where $\eta, \alpha \in L$ such that $\eta < \alpha$ and α is a dual atom in L.

Then $\psi_{a,b}(a_\alpha \vee a_\eta) = \psi_{a,b}(a_\alpha) = g_{\alpha,b}$

and $\psi_{a,b}(a_\alpha) \vee \psi_{a,b}(a_\eta) = g_{\alpha,b} \vee \underline{1} = \underline{1}$.

$\Rightarrow \psi_{a,b}(a_\alpha \vee a_\eta) \neq \psi_{a,b}(a_\alpha) \vee \psi_{a,b}(a_\eta)$.

Remark 4.4. *Let X be any nonempty set and L be a finite chain with the atom α and the dual atom β. For any $x, y \in X$, define $c_{x,y} : L^X \to L^X$ by*

$$c_{x,y}(f) = \begin{cases} \underline{0} & if \quad f = \underline{0} \\ g_{y,\beta} & if \quad f = x_\alpha \\ \underline{1} & \quad otherwise \end{cases},$$

where $g_{y,\beta} \in L^X$ is defined as $g_{y,\beta}(y) = \beta$ and $g_{y,\beta}(z) = 1, \forall z (\neq y) \in X$. Clearly,

$g_{y,\beta}$ is a dual atom in L^X.

It can be easily checked that $c_{x,y}$ is an L-closure operator and $g_{y,\beta}$ can be replaced by any dual atom in L^X. Therefore the number of such L-closure operators is $|X|^2$.

Theorem 4.5. *Let X be a nonempty set and L be a chain with the atom α and the dual atom β. An L-closure operator is an infra L-closure operator if and only if it is of the form $c_{x,y}$ for some $x, y \in X$.*

Proof: Let c be any L-closure operator on X such that

$$c \le c_{x,y} \Rightarrow c_{x,y}(f) \le c(f), \forall f \in L^X.$$

Therefore $c(f) = \underline{1}, \forall f (\ne x_\alpha) \in L^X$ and $g_{y,\beta} \le c(x_\alpha)$. Since $g_{y,\beta}$ is a dual atom in L^X, it follows that either $c(x_\alpha) = g_{y,\beta}$ or $c(x_\alpha) = \underline{1}$.

If $c(x_\alpha) = g_{y,\beta}$, then $c = c_{x,y}$ and if $c(x_\alpha) = \underline{1}$, then $c = I$.

Hence $c_{x,y}$ is an infra L-closure operator.

Conversely, suppose that c is any infra L-closure operator in $LC(X)$. Then c must be of the form

$$c(f) = \begin{cases} \underline{0} & if \quad f = \underline{0} \\ \ne \underline{1} & if \quad f = g \\ \underline{1} & \text{for all} \quad f(\ne \underline{0}, g) \in L^X \end{cases} \quad \text{for some } g \in L^X .$$

If $\exists$ an L-subset $h(\ne \underline{0}) \in L^X$ such that $h < g$, then $h \vee g = g$ and

$c(h \vee g) = c(g)$.

$\Rightarrow c(h) \vee c(g) = c(g)$

$\Rightarrow \underline{1} \vee c(g) = c(g)$

$\Rightarrow c(g) = \underline{1}$, a contradiction.

$\Rightarrow g$ must be an atom in L^X i.e. $g = x_\alpha$ for some $x \in X$ and for the atom $\alpha \in L$.

Now, if $c(g) < f$ for some $f(\ne \underline{1}) \in L^X$, then $c_1 : L^X \to L^X$ defined by :

$$c_1(h) = \begin{cases} \underline{0} & if \quad h = \underline{0} \\ f & if \quad h = x_\alpha \\ \underline{1} & \quad otherwise \end{cases}$$

is an L-closure operator such that $c_1 \ne I$ and $c_1 < c$, a contradiction. Therefore $c(g)$ must be a dual atom in L^X i.e. $c(g)(t) = \underline{1}, \forall t \in X$ except for some $y \in X$ and $c(g)(y) = \beta$. $\Rightarrow c = c_{x,y}$.

Thus all infra L-closure operators are of the form $c_{x,y}$ for some $x, y \in X$.

Remark 4.6. *If L is a chain with the atom α and the dual atom β, then for any nonempty set X, there are $|X|^2$ infra L-closure operators in $LC(X)$.*

Remark 4.7. *Let $X=\{x,y,z\}$ and $L=\{0,\alpha,\beta,1\}$ be a chain with $0<\alpha<\beta<1$. Then there are 9 infra L-closure operators given by*

$$c_{x,y}(f)=\begin{cases}\underline{0} & if \quad f=\underline{0}\\ g_{y,\beta} & if \quad f=x_\alpha\\ \underline{1} & \quad otherwise\end{cases}$$

where $x,y\in X$ and $g_{y,\beta}\in L^X$ is defined as $g_{y,\beta}(y)=\beta$ and $g_{y,\beta}(x)=g_{y,\beta}(z)=1$. It can be easily checked that the L-closure operator $c:L^X\to L^X$ defined by

$$c(f)=\begin{cases}\underline{0} & if \quad f=\underline{0}\\ \underline{\alpha} & if \quad f=x_\beta\\ \underline{1} & \quad otherwise\end{cases}$$

can not be written as the join of infra L-closure operators. Thus in general $LC(X)$ is not an atomic lattice.

5. Ultra L-closure operators

Definition 5.1. *An L-topology F on X is called an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.*

Definition 5.2. *An L-closure operator on X is called an ultra L-closure operator if the only L-closure operator on X strictly larger than it, is D.*

Theorem 5.3. *Let c_1 and c_2 be two L-closure operators such that $c_1\le c_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are the L-topologies associated with the L-closure operators c_1 and c_2 respectively, then $\mathcal{F}_1\subseteq\mathcal{F}_2$.*

Proof: Let $g\in\mathcal{F}_1$.

$\Rightarrow c_1(g')=g'$, where g' is complement of g.

$\Rightarrow g'\le c_2(g')\le c_1(g')=g'$.

$\Rightarrow c_2(g')=g'$.

$\Rightarrow g\in\mathcal{F}_2\Rightarrow\mathcal{F}_1\subseteq\mathcal{F}_2$.

Theorem 5.4. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two L-topologies such that $\mathcal{F}_1\subseteq\mathcal{F}_2$. If c_1 and c_2 are the L-closure operators associated with the L-topologies $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, then $c_1\le c_2$.*

Proof: For any $f\in L^X$, let $\bar{f}_1$ and $\bar{f}_2$ be the closure of f in the L-topological

spaces $(L^X, \mathcal{F}_1)$ and $(L^X, \mathcal{F}_2)$ respectively.
Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$,
$\Rightarrow \bar{f}_2 \leq \bar{f}_1$.
$\Rightarrow c_2(f) \leq c_1(f)$.
$\Rightarrow c_1 \leq c_2$.

Theorem 5.5. *Let c be an ultra L-closure operator. If $\mathcal{F}$ is the L-topology associated with the L-closure operator c, then $\mathcal{F}$ is an ultra L-topology.*
Proof: If $\mathcal{F}$ is not an ultra L-topology, then there exists an L-topology $\mathcal{F}_1$ such that $\mathcal{F}_1 \neq L^X$ and $\mathcal{F} \subset \mathcal{F}_1$. Let c_1 be the L-closure operator associated with the L-topology $\mathcal{F}_1$.Then $c_1 \neq D$.
Since $\mathcal{F} \subset \mathcal{F}_1$,
$\Rightarrow c \leq c_1$ and $\exists$ an L-subset $g \in L^X$ such that $g \in \mathcal{F}_1$ but $g \notin \mathcal{F}$.
$\Rightarrow c_1(g') = g'$ and $c(g') \neq g'$
$\Rightarrow c < c_1$, a contradiction.
Hence $\mathcal{F}$ is an ultra L-topology.

Theorem 5.6. *Let $\mathcal{F}$ be an ultra L-topology. If c is the L-closure operator associated with the L-topology $\mathcal{F}$, then c is an ultra L-closure operator.*
Proof: Suppose, there exists an L-closure operator c_1 such that $c \leq c_1$. Let $\mathcal{F}_1$ be the L-topology associated with the L-closure operator c_1.
Since $c \leq c_1 \Rightarrow \mathcal{F} \subseteq \mathcal{F}_1$ and $\mathcal{F}$ is an ultra L-topology so it follows that either $\mathcal{F}_1 = \mathcal{F}$ or $\mathcal{F}_1 = L^X =$ discrete L-topology.
If $\mathcal{F}_1 = \mathcal{F}$, then $c_1 = c$ and if $\mathcal{F}_1 = L^X$, then $c_1 = D =$ discrete L-closure operator.
Hence c is an ultra L-closure operator.

Theorem 5.7. *Let X be a nonempty set and L be a bounded chain. Then an L-closure operator is an ultra L-closure operator if and only if it is the L-closure operator associated with some ultra L-topology on X.*

Remark 5.8. *If $L = [0,1]$, then there is no ultra L-topology in L^X and hence no ultra L-closure operator in $LC(X)$* [4].

Remark 5.9. *Let $X = \{x, y, z\}$ and $L = \{0, \alpha, \beta, 1\}$ be a chain with* $0 < \alpha < \beta < 1$.
Then the L-closure operator $c : L^X \to L^X$ defined by

$$c(f)=\begin{cases}\underline{0} & if \quad f=\underline{0}\\ \underline{\alpha} & if \quad f=x_\beta\\ \underline{1} & otherwise\end{cases}$$

has no complement $\Rightarrow LC(X)$ is not complemented in general.

Remark 5.10. [5] *If* $L=[0,1]$, *then* $LC(X)$ *is not complemented.*

6. Conclusion

In this paper, we have identified infra L-closure operators and established a relation between ultra L-closure operators and ultra L-topologies. Also it is proved that $LC(X)$ is a distributive lattice when L is a bounded chain. Lattice of L-closure operators when L is a bounded lattice other than a chain, will be discussed in future papers.

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