Annals of Pure and Applied Mathematics
Vol. 17, No. 2, 2018, 151-155
ISSN: 2279-087X (P), 2279-0888(online)
Published on 25 May 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v17n2a1

Annals of **Pure and Applied Mathematics**

1-Modular Dual Nearlattices

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Received 6 April 2018; accepted 22 May 2018

Abstract. Jayaram [2] introduced the concept of 0-modular semilattice. Recently, Rahman et al. [4] have introduced the concept of 0-modular Nearlattice. In this paper, we discuss 1-modular dual nearlattice. A dual nearlattice S with 1 is said to be 1-modular if for all $a, b, c \in S$ with $c \geq a$ and $a \vee b = 1$ imply $a \vee (b \wedge c) = c$ provided $b \wedge c$ exists. Akhter and Noor [8] have discussed 1-distributive join semilattice. In this paper, we include several characterizations of 1-modular dual nearlattices.

Keywords: 1-distributive join semilattice, 1-modular dual nearlattice, prime filter, join prime element, dual atom

***AMS Mathematics Subject Classification (2010)*:** 06A12, 06A99, 06B10

1. Introduction

Varlet [3] introduced the concept of 0-distributive and 0-modular lattices. A lattice L with 0 is called 0-distributive if for all $a, b, c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. A lattice L is called 0-modular if all $a, b, c \in L$ with $c \leq a$ and $a \wedge b = 0$ imply $a \wedge (b \vee c) = c$. Of course, every distributive lattice is both 0-distributive and 0-modular. [1,2,5,6] have studied different properties of 0-distributivity and 0-modularity in lattices and semilattices. Akhter et al. [7] discussed some properties of 1-distributive join-semilattice. A join-semilattice S with 1 is called 1-distributive if for all $a, b, c \in S$ with $a \vee b = 1 = a \vee c$ imply $a \vee d = 1$ for some $d \leq b, c$. In this paper, we discuss 1-modular dual nearlattice and give several nice characterizations of 1-modular dual nearlattice. A dual nearlattice S is a join-semilattice together with the property that any two elements possessing a common lower bound, have a infimum. A dual nearlattice S with 1 is called 1-modular if for all $a, b, c \in S$ with $c \geq a$ and $a \vee b = 1$ imply$a \vee (b \wedge c) = c$ provided $b \wedge c$ exists.

A lattice L with 1 is called 1-distributive if for all $a, b, c \in L$ with $a \vee b = a \vee c = 1$ imply $a \vee (b \wedge c) = 1$. A lattice L with 1 is called1-modular if for all $a, b, c \in L$ with $c \geq a$ and $a \vee b = 1$ imply $a \vee (b \wedge c) = c$. A lattice L ith 0 is called semi complemented if for any $a \in L, (a \neq 1)$ there exists $b \in L, (b \neq 0)$ such that $a \wedge b = 0$. Dually a lattice L with 1 is called dual semi complemented if for any$a \in L, (a \neq 0)$ there exists $b \in L$, $(b \neq 1)$ such that $a \vee b = 1$. A lattice L with 0 and 1 is called complemented if for any $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. A lattice L with 0 is called weakly complemented if for any distinct elements $a, b \in L$, there exists $c \in L$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$ (or vice versa).

Let S be a dual nearlattice. A non-empty subset F of S is called filter if
(i) $a, b \in F$ implies there exists $d \leq a, b$ such that $d \in F$
(ii) $a \in F, x \in S$ with $x \geq a$ implies $x \in F$.

A filter F is called proper filter of a dual nearlattice S if $F \neq S$. A proper filter F in S is called prime filter if $a \vee b \in F$ implies $a \in F$ or $b \in F$. For $a \in S$, the filter $F = \{x \in S | x \geq a\}$ is called the principal filter generated by a. It is denoted by $[a)$. Let S be a dual nerlattice. A subset I of S is called an ideal if (i) $a, b \in I$ implies $a \vee b \in I$ (ii) $a \in S, i \in I$ with $a \leq i$ implies $a \in I$.

An ideal I of a dual nearlattice S is called prime ideal if $I \neq S$ and $S - I$ is prime filter.

An element a of a nearlattice S is called meet prime if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. An element a of a dual nearlattice S is called join prime if $b \vee c \geq a$ implies either $b \geq a$ or $c \geq a$. A non-zero element a of a lattice L with 0 is an atom if for any $b \in L$ with $0 \leq b \leq a$ implies either $0 = b$ or $b = a$. An element a of a dual nearlattice S with 1 is called a dual atom if for any $b \in S$ with $a \leq b \leq 1$ implies $a = b$ or $b = 1$.

2. Main results

To obtain the main results of this paper we need to prove the following first three theorems and one corollary.

Theorem 1. A dual nearlattice S with 1 is 1-modular if and only if for all $a, b, c \in S$ with $c \geq a$, $a \vee b = 1, a \wedge b = c \wedge b$ imply $a = c$ provided $a \wedge b$ exists.

Proof: Suppose S is 1-modular and $a, b, c \in S$ with $c \geq a$, $a \vee b = 1$. Also let $a \wedge b = c \wedge b$. If $a \wedge b$ exists then $c \wedge b$ exists by the lower bound property. Then $a = a \vee (a \wedge b) = a \vee (b \wedge c) = c$. Conversely, let $a, b, c \in S$ with $c \geq a, a \vee b = 1$ and $b \wedge c$ exists. Also let $a \wedge b = c \wedge b$ implies $a = c$. Here $c \geq a \vee (b \wedge c)$ and $b \vee [a \vee (b \wedge c)] = b \vee a = 1$. Now, $a \vee (b \wedge c) \geq b \wedge c$, so $b \wedge [a \vee (b \wedge c)] \geq (b \wedge c)$. Also $c \geq a \vee (b \wedge c)$ implies $b \wedge [a \vee (b \wedge c)] \leq (b \wedge c)$ and so $b \wedge c = b \wedge [a \vee (b \wedge c)]$, so by the given conditions $c = a \vee (b \wedge c)$ which implies S is 1-modular.

Theorem 2. A dual nearlattice S with 1 is 1-modular if and only if the interval $[x, 1]$ for each $x \in S$ is 1-modular.

Proof: If S is 1-modular then trivially $[x, 1]$ is 1-modular for each $x \in S$. Conversely, let $[x, 1]$ is 1-modular for each $x \in S$. Let $a, b, c \in S$ with $a \vee b = 1, c \geq a$ and $b \wedge c$ exists. Choose $t = b \wedge c$. Then $a \vee (b \wedge c) = a \vee [(t \vee b) \wedge (t \vee c)] = (t \vee a) \vee [(t \vee b) \wedge (t \wedge c)] = t \vee c = c$ as the interval $[t, 1]$ is 1-modular.

Corollary 3. A dual nearlattice S with 1 is 1-distributive if and only if the interval $[x, 1]$ for each $x \in S$ is 1-distributive.

Theorem 4. Let S be a dual nearlattice. Then the intersection of any two filters of S is also a filter.

Proof: Let F, G be two filters of a dual nearlattice S. Let $a \in F \cap G$ and $b \in S$ with $b \geq a$. Then $a \in F$ and $a \in G$. Since both F and G are filters, so $b \in F$ and $b \in G$. Hence $b \in F \cap G$.

Again let $a, b \in F \cap G$. So $a, b \in F$ and $a, b \in G$. Since F and G are both filters, then there exist $f \in F$ and $g \in G$ such that $f, g \leq a, b$. Let $c = f \wedge g$. Then $c \in F \cap G$, where $c \leq a, b$. Hence, $F \cap G$ is a filter.

Theorem 5. For a dual nearlattice S with 1, if $I(S)$ is 1-modular, then S is 1-modular.

Proof: Suppose $I(S)$ is 1-modular. Let $a, b, c \in S$ with $a \vee b = 1, c \geq a$ and $b \wedge c$ exists. Then $(a] \vee \left((b] \wedge (c]\right) = (c]$ as $I(S)$ is 1-modular. Thus $(a \vee (b \wedge c)] = (c]$ and so $a \vee (b \wedge c) = c,$ which implies that S is 1-modular.

Theorem 6. A dual nearlattice S with 1 is 1-modular if and only if the lattice of filters of the interval $[x, 1]$ for each $x \in S$ is 0-modular.

Proof: Let S be 1-modular. Choose any $x \in S$. Then $[x, 1]$ is also 1-modular. Let F, G, H be filters of the lattice $[x, 1]$ such that $F \supseteq H$ and $F \cap G = [1)$. Then $F \cap (G \vee H) \subseteq H$ is obvious. Let $h \in H$. Now $F \cap G = [1)$ implies $1 = f \vee g$ for some $f \in F$ and $g \in G$. Thus $h \vee f \geq f$ and $f \vee g = 1$ imply $f \vee [g \wedge (h \vee f)] = h \vee f$ as S is 1-modular. So $h \vee f \in F \cap (G \vee H)$ and hence $h \in F \cap (G \vee H)$. Therefore, $F \cap (G \vee H) = H$ and so the lattice of filters of $[x, 1]$ is 0-modular.

Conversely, suppose the lattice of filters of $[x, 1]$ is 0-modular. Let $a, b, c \in [x, 1]$such that $c \geq a,\ a \vee b = 1.$ Then $[a) \supseteq [c)$ and $[a) \wedge [b) = [1)$. So by 0-modular property, $[a) \wedge \left([b) \vee [c)\right) = [c)$. Thus, $\left[a \vee (b \wedge c)\right) = [c)$ and so$a \vee (b \wedge c) = c$. This implies $[x, 1]$ is1-modular. Therefore, by theorem 2, S is 1-modular.

Theorem 7. If a dual nearlattice S with 1 is 1-distributive and $[x, 1]$ is dual semi complemented for each $x \in S$, then the interval $[x, 1]$ is0-distributive for each $x \in S$.

Proof: Let $a, b, c \in [x, 1]$ with $a \wedge b = x = a \wedge c$. Suppose $a \wedge (b \vee c) \neq x$. Then there exists $p \neq 1$ in $[x, 1]$ such that $p \vee \left(a \wedge (b \vee c)\right) = 1$. Then $a \vee p = 1 = (b \vee c) \vee p$. Thus $p \vee b \vee a = 1 = (p \vee b) \vee c$. This implies $(p \vee b) \vee (a \wedge c) = 1$ as S is 1-distributive. This implies $1 = p \vee b \vee x = p \vee b$. Again, using the 1-distributivity of S, $p \vee (a \wedge b) = 1$. That is, $1 = p \vee x = p$ which gives a contradiction. Therefore, $a \wedge (b \vee c) = x$. Hence, $[x, 1]$ is 0-distributive.

Theorem 8. Let S be a 1-modular dual nearlattice and I, J are two ideals such that $I \vee J = (1]$ and $I \cap J = (x]$ for some$x \in S$. Then both I and J are principal ideals.

Proof: Suppose $I \vee J = (1]$ and $I \cap J = (x]$ for some $x \in S$. Then $1 = i \vee j$ for some $i \in I$ and $j \in J$. Let $b = x \vee i$ and $c = x \vee j$. Then $b \in I$ and $c \in J$. We claim that $I = (b]$

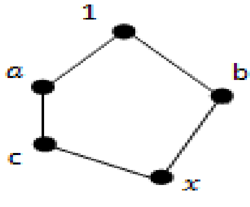

Figure 1: Pentagonal sublattice

and $J = (c]$. Indeed, if for instance, $J \neq (c]$, then there exists $a \in J$ such that $a > c$. Then $\{x, c, a, b, 1\}$ is a pentagonal sublattice of S. This implies S is not 1-modular and which gives a contradiction.
Therefore, $J = (c]$. Similarly, $I = (b]$. Hence both I and J are principal ideals.
A dual nearlattice S with 1 is called a dual semi Boolean lattice if it is distributive and the interval $[x, 1]$ for each $x \in S$ is complemented.

Theorem 9. If a section complemented 1-modular dual nearlattice S is1-distributive, then it is dual semi Boolean.
Proof: Let $b < a$ for some $a, b \in S$. Then $b < a \leq 1$. Since $[b, 1]$ is complemented so there exists $c \in [b, 1]$ such that $c \wedge a = b$ and$c \vee a = 1$. Now, if $b \vee c = 1$, then by the 1-modularity of S, $b = b \vee (c \wedge a) = a$, which is a contradiction. Therefore, $b \vee c \neq 1$. This implies S is weakly complemented. Also since S is 1-distributive, so by Corollary 3, $[x, 1]$ is Boolean for each $x \in S$ and so S is dual semi Boolean.

Lemma 10. In a bounded dual semi complemented lattice L, every join prime element is an atom.
Proof: Suppose a is a join prime element. Let $0 < b \leq a$. Then $0 < b \leq 1$. Since L is dual semi complemented, there exists $c \in L(c \neq 1)$ such that $b \vee c = 1$. Since $b \leq a$, so $c \vee a = 1$. Since a is join prime element so this implies $c \geq a$ or $b \geq a$. But $c \geq a$ implies $c = c \vee a = 1$, which is a contradiction. Hence $b \geq a$ and so $a = b$. Therefore, a is an atom.

Lemma 11. Let L be a bounded 1-modular lattice. If $b \in L$ is an atom and $a \vee b = 1$ for some $a \neq 0 (a \in L)$, then a is a dual atom.
Proof: Suppose $a \leq c < 1$ for some $c \in L$. Since $c \geq a$ and $a \vee b = 1$, so by 1-modularity, $a \vee (b \wedge c) = c$. Also since $c < 1$, it follows that $b > b \wedge c$ and so $b \wedge c = 0$ as b is an atom. Consequently, $a = a \vee 0 = a \vee (b \wedge c) = c$ by 1-modularity. Thus a is a dual atom.

Lemma 12. Let S be a 1-modular dual nearlattice and $[x, 1]$ is dual semi complemented for each $x \in S$. If for each $x \in S$, 1 is the join of a finite number of join prime elements in $[x, 1]$. Then x is the meet of finite number of dual atoms in $[x, 1]$.
Proof: Let $1 = \vee_{i=1}^{n} p_i$, where $p_i's$ are join prime elements in $[x, 1]$. By Lemma 10, each p_i is an atom in $[x, 1]$. Since each $p_i \neq 1$ and $[x, 1]$ is dual semi complemented, so there exists $q_i \in [x, 1]$ such that $p_i \vee q_i = 1$, $i = 1,2, \cdots, n$. Also by Lemma 11, each q_i is a dual atom in $[x, 1]$. Let$c = \wedge_{i=1}^{n} q_i$. Then $c \wedge p_i = x$ as p_i is an atom for each i. As $[x, 1]$ is dual semi complemented and 1 is the join of finite number of join primes, hence $[x, 1]$ is 1-distributive and so by Theorem 5, $[x, 1]$ is 0-distributive.Therefore, $c \wedge (\vee_{i=1}^{n} p_i) = x$. That is, $c = c \wedge 1 = x$. Thus $x = \wedge_{i=1}^{n} q_i$.

We conclude this paper with the following Theorem which trivially follows from [2].

Theorem 13. For a dual nearlattice S with 1, S is dual semi Boolean if and only if the following conditions are satisfied.

(i) $[x, 1]$ is dual semi complemented for each $x \in S$

(ii) S is 1-modular

(iii) 1 is the join of a finite number of join primes

3. Conclusion

In this paper, we study the concept of 1-modular dual nearlattice. We also include several characterizations of 1-modular dual nearlattices and prove some results on 1-modular dual nearlattices. Here we prove that, a dual nearlattice S with 1, is 1-modular if and only if the lattice of filters of the interval $[x, 1]$ for each $x \in S$ is 0-modular.

Acknowledgements. The authors are grateful to the reviewers for their valuable comments and suggestions for the improvement of the paper.

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