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A Size Multipartite Ramsey Problem Involving the Claw Graph

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Abstract. Let $K_{j\times s}$ denote a complete balanced multipartite graph consisting of j partite sets of uniform size s. For any two colouring of the edges of a graph $K_{j\times s}$, we say that $K_{j\times s} \rightarrow (K_{1,3},G)$, if there exists a copy of $K_{1,3}$(Claw graph) in the first colour or a copy of G in the second colour. $m_j(K_{1,3},G)$ is defined as the smallest positive integer s such that $K_{j\times s} \rightarrow (K_{1,3},G)$. In this paper we find all such $m_j(K_{1,3},G)$ for all graphs G on 4 vertices.

Keywords: Ramsey theory, Multipartite Ramsey numbers

AMS Mathematics Subject Classification (2010): 05C55, 05D10

1. Introduction

Given any two graphs G and H, the classical Ramsey number (see [2,4,7,8]) $r(H,G)$ is defined as the smallest positive integer n such that $K_n \rightarrow (H,G)$. A natural generalization of the popular classical Ramsey number is the size multipartite Ramsey number which was introduced a few decades ago (see [1, 9]). The balanced complete multipartite graph denoted by $K=K_{j\times s}$ is defined as a graph consisting of j uniform partite sets s, where

$$V(K) = \{v_{1,1}, v_{1,2}, \ldots, v_{1,s}, v_{2,1}, v_{2,2}, \ldots, v_{2,s}, \ldots, v_{j,1}, v_{j,2}, \ldots, v_{j,s}\}$$

and $E(K) = \bigcup_{1\le m,m'\le j} \{(v_{m,i}, v_{m',i'}) \mid 1 \le i,\, i' \le s, \text{ and } m \ne m'\}$. Given any two colouring of the edges of the graph K with H_R and H_B representing the red and blue subgraphs of K, we say that $K \rightarrow (K_{1,3},G)$, if there exists a red copy of $K_{1,3}$ in H_R or a copy of G in H_B. The size Ramsey multipartite number $m_j(K_{1,3},G)$ is defined as the smallest natural number s such that $K_{j\times s} \rightarrow (K_{1,3},G)$. In this paper we exhaustively find $m_j(K_{1,3},G)$ for all graphs G on 4 vertices.

2. Notation

Given any two colouring of the edges of the graph $K = K_{j\times s}$, let the red and blue subgraphs of K with $V(K) = V(H_R) = V(H_B)$ be denoted by H_R and H_B respectively. In such a situation, we say that $K \rightarrow (K_{1,3},G)$, if there exists a red copy of $K_{1,3}$ in H_R or a blue copy of G in H_B. We define red neighbourhood of any vertex $v \in K$ as the set of vertices adjacent to v in red and is denoted by $N_R(v)$. We also define the red degree of any vertex $v \in K$ as $|N_R(v)|$. Define $\Delta(H_R)$ ($\delta(H_R)$) be the maximum (minimum) degree of the vertices of H_R. It is worth noting that any two colouring of $K_{j\times s}$ with H_R containing no $K_{1,3}$ will satisfy $\delta(H_B) \geq s(j-1)-2$. The summary of our findings is illustrated in the following table.

$m_j(T,G)$	$j=$	3	4	5	6	7	8	9	≥10
Row 1	$4K_1$	2	1	1	1	1	1	1	1
Row 2	$P_2 U 2K_1$	2	1	1	1	1	1	1	1
Row 3	$2K_2$	2	2	1	1	1	1	1	1
Row 4	$P_3 U K_1$	2	2	1	1	1	1	1	1
Row 5	P_4	3	2	1	1	1	1	1	1
Row 6	$K_{1,3}$	3	2	2	1	1	1	1	1
Row 7	$C_3 U K_1$	3	3	2	2	1	1	1	1
Row 8	C_4	3	2	2	1	1	1	1	1
Row 9	$K_{1,3}+x$	3	3	2	2	1	1	1	1
Row 10	B_2	4	3	2	2	1	1	1	1
Row 11	K_4	∞	4	3	3	2	2	2	1

Table 1: Values of $m_j(T,G)$.

The next section deals with finding $m_j(K_{1,3},G)$ the entries of the above table. Clearly the rows corresponding to row 1, row 2, row 4, row 5, follows from Syafrizal et al. (see [3, 7, 9]) and row 7 and row 10 follows from Jayawardene et al. (see [5, 6]).

3. Size Ramsey numbers $m_j(K_{1,3},G)$ when G is a connected proper subgraph on K_4

Theorem 1. *If* $j \geq 3$, *then*

$$m_j(K_{1,3},K_4) = \begin{cases} 1 & j \geq 10 \\ 2 & j \in \{7,8,9\} \\ 3 & j \in \{5,6\} \\ 4 & j = 4 \\ \infty & j = 3 \end{cases}$$

Proof: Since (see [2]), when $j \geq 10$, we get $m_j(K_{1,3}, K_4) = 1$.

For $j \in \{7,8,9\}$, consider the graph $K_{9\times 1}$ such that $H_R = 3K_3$ and $H_B = K_{3,3,3}$. Then the graph has no red $K_{1,3}$ and has no blue K_4. Therefore, $m_9(K_{1,3},K_4) \geq 2$. Next to show $m_7(K_{1,3},K_4) \leq 2$, consider any red and blue colouring of $K_{7\times 2}$, such that H_R contains no red $K_{1,3}$ and H_B contains no blue K_4. From [5] there is a blue C_3 in H_B as $m_7(K_{1,3},C_3) \leq 2$. Without loss of generality assume that the blue C_3 is induced by say $v_{1,1},v_{2,1},v_{3,1}$. Let $W=\{v_{k,i} \mid 1 \leq i \leq 2$, 4

$\leq k \leq 7$ }. In order to avoid a blue K_4, every single vertex in W has to be adjacent to some vertex of S in red. Then by pigeon hole principle at least three vertices of W have to be adjacent to some vertex $s \in S$. That is, $s \in S$ will be the root of a red $K_{1,3}$, a contradiction. Hence, $m_7(K_{1,3},K_4) \leq 2$. Therefore, we get $2 \leq m_9(K_{1,3},K_4) \leq 2 \leq m_8(K_{1,3},K_4) \leq m_7(K_{1,3},K_4) \leq 2$. That is, $m_j(K_{1,3},K_4) = 2$ for $j \in \{7,8,9\}$.

For $j \in \{4,5,6\}$, consider the graph $K_{6\times2}$, such that $H_R = 3C_4$ as illustrated in Figure 1. Then the graph H_R has no red $K_{1,3}$ and has no blue K_4. Therefore we get, $m_6(K_{1,3},K_4) \geq 3$. Next to show $m_5(K_{1,3},K_4) \leq 3$, consider any red and blue colouring of $K_{5\times3}$ such that H_R contains no red $K_{1,3}$ and H_B contains no blue K_4. As $m_5(K_{1,3},B_2) \leq 3$ from [6] there is a blue B_2 in H_B. As H_R has no blue K_4, without loss of generality assume that the blue B_2 is induced by say $v_{2,1},v_{3,1},v_{4,1},v_{5,1}$ such that solitary red edge among these vertex is given by $(v_{2,1},v_{3,1})$. Let $S = \{v_{3,1}, v_{4,1}, v_{5,1}\}$ and let $W = \{ v_{k\,i} \mid 1 \leq i \leq 3,\ 1 \leq k \leq 2 \}$. In order to avoid a blue K_4, every single vertex in W has to be adjacent to a vertex of S in red. Thus, as there is no red $K_{1,3}$, without loss of generality each of the three vertices of S will be adjacent in red to exactly two vertices of W with the added condition that $(v_{2,1},v_{3,1})$ is red.

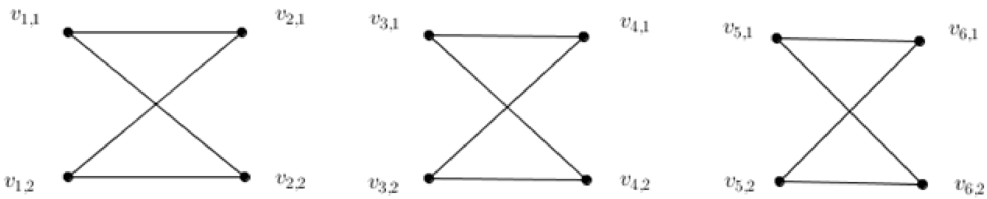

Figure 1: H_R graph related to the proof of $m_6(K_{1,3}, K_4) \geq 3$

However, for$\{v_{2,1},v_{3,1},v_{4,1},v_{5,1}\}$ not to induce a blue K_4 graph, the edge $(v_{2,1},v_{3,2})$ has to be a red edge (as in order to avoid a red $K_{1,3}$, both $v_{4,1}$ and $v_{5,1}$ cannot be adjacent to any vertices of outside of W in red). Similarly, in order for$\{v_{2,1},v_{3,3},v_{4,1},v_{5,1}\}$ not to induce a blue K_4 graph, the edge $(v_{2,1},v_{3,3})$ has to be a red edge. Thus, we get that $\{v_{2,1},v_{3,1},v_{3,1},v_{3,2},v_{3,3}\}$ will induce a red $K_{1,3}$, a contradiction. Therefore, $m_5(K_{1,3},K_4) \leq 3$. Therefore, we get $3 \leq m_6(K_{1,3},K_4) \leq m_5(K_{1,3},K_4) \leq 3$. That is, $m_5(K_{1,3},K_4) = 3$ for $j \in \{5,6\}$

Next let us deal with the case $j = 4$. Consider the colouring of $K_{4\times3}$, generated by $H_R = 3C_4$ as shown in Figure 2. Then, $K_{4\times3}$ will not contain a red $K_{1,3}$ as H_R is a regular graph of red degree 2.

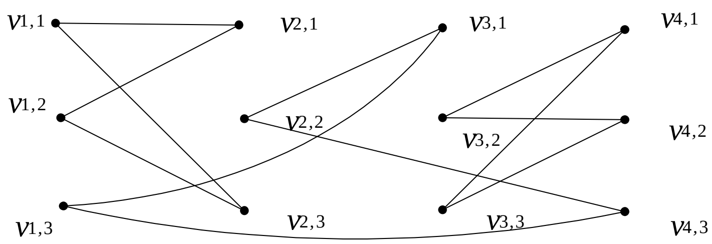

Figure 2: H_R graph related to the proof of $m_4(K_{1,3},K_4) \geq 4$

Claim. H_B is a regular graph containing no blue K_4.
Proof of Claim. In order to have a blue K_4, each partite set must contain exactly one vertex of the K_4. Suppose that H_B contains a blue K_4 denoted by H. Then, $V(H)$ will consist of four vertices x_1,x_2,x_3 and x_4, such that x_i, $i \in \{1,2,3,4\}$ belongs to the i^{th} partite set.

Case 1. If $x_1 = v_{1,1}$ or $v_{1,2}$.
Then x_2 will be forced to be equal to $v_{2,2}$. Then the only options left for x_3 will be $v_{3,2}$ or $v_{3,3}$. However, either one of these two choices will not leave an option for x_4, a contradiction.

Case 2. If $x_1 = v_{1,3}$.
Then x_4 will be forced to be equal to $v_{4,1}$ or $v_{4,2}$ However, either one of these two choices will not leave an option for x_3, a contradiction.

Therefore, in $K_{4\times 3}$, H_R contains no red $K_{1,3}$ and H_B contains no blue K_4. Thus, we get $m_3(K_{1,3},K_4) \geq 4$. Next to show, $m_4(K_{1,3},K_4) \leq 4$ consider any red and blue colouring of $K_{4\times 4}$, such that H_R contains no red $K_{1,3}$ and H_B contains no blue K_4. As $m_4(K_{1,3},B_2) \leq 4$ from [6] we get that there is a blue B_2, in H_B. As H_R has no blue K_4, without loss of generality assume that the blue B_2 is induced by say $v_{1,1},v_{2,1},v_{3,1},v_{4,1}$ such that the solitary red edge among these vertex is given by $(v_{1,1},v_{2,1})$. Define $S = \{v_{2,1},v_{3,1},v_{4,1}\}$, $S_1 = \{v_{1,2},v_{1,3},v_{1,4}\}$, $S_2 = \{v_{2,2},v_{2,3},v_{2,4}\}$ and $S_3 = \{v_{1,1},v_{3,1},v_{4,1}\}$. Next, in order to avoid a blue K_4, every single vertex in S_1 has to be adjacent to a vertex of S in red. Without loss of generality, this gives rise to the following three cases.

Case I. $(v_{1,2},v_{2,1})$,$(v_{1,3},v_{3,1})$ and $(v_{1,4},v_{3,1})$ are red edges.
In order to avoid a blue K_4, every single vertex in S_2 has to be adjacent to a vertex of S_3 in red. Thus, without loss of generality, we get the following graph represented in Figure 3.

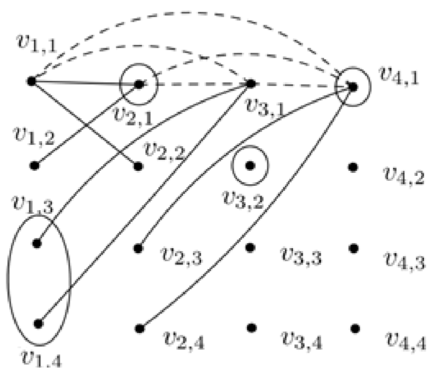

Figure 3:

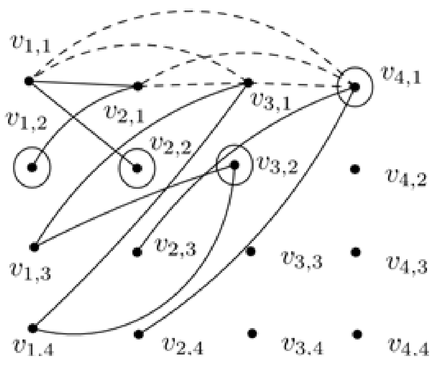

Figure 4:

In order for $\{v_{1,3},v_{2,1},v_{3,2},v_{4,1}\}$ not to induce a blue K_4, $(v_{1,3},v_{3,2})$ has to be a red edge. Similarly, in order for $\{v_{1,4},v_{2,1},v_{3,2},v_{4,1}\}$ not to induce a blue K_4, $(v_{1,4},v_{3,2})$ has to be a red edge. This gives rise to Figure 4. As indicated in Figure 4, in order for $\{v_{1,2},v_{2,2},v_{3,2},v_{4,1}\}$ not to induce a blue K_4, $(v_{1,2},v_{2,2})$ has to be a red edge. Finally, in order for $\{v_{1,1},v_{2,3},v_{3,2},v_{4,2}\}$, $\{v_{1,1},v_{2,3},v_{3,2},v_{4,3}\}$ and $\{v_{1,1},v_{2,3},v_{3,2},v_{4,4}\}$ not to induce a blue K_4, $(v_{2,3},v_{4,2})$, $(v_{2,3},v_{4,3})$ and $(v_{2,3},v_{4,4})$, has to be red edges. That is $\{v_{2,3},v_{4,2},v_{4,3},v_{4,4}\}$ will induce a red $K_{1,3}$, a contradiction.

Case II. $(v_{1,2},v_{2,1})$,$(v_{1,3},v_{3,1})$ and $(v_{1,4},v_{4,1})$ are red edges.
In order to avoid a blue K_4, every single vertex in S_2 has to be adjacent to a vertex of S_3 in red. Thus, without loss of generality we get the following graph represented in Figure 5.

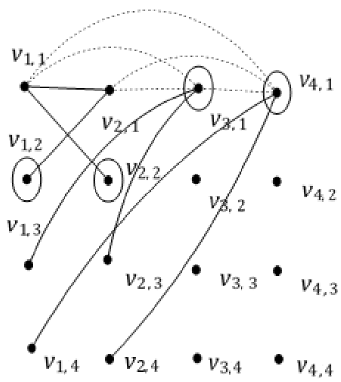

Figure 5:

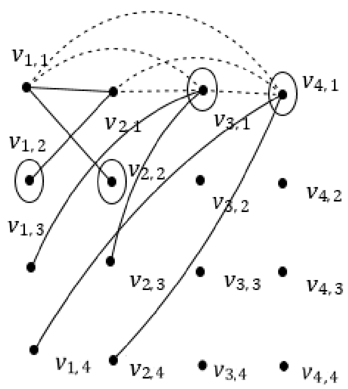

Figure 6:

In order for $\{v_{1,2},v_{2,2},v_{3,1},v_{4,1}\}$ not to induce a blue K_4, $(v_{1,2},v_{2,2})$ has to be a red edge. This will result in the graph represented in Figure 6.

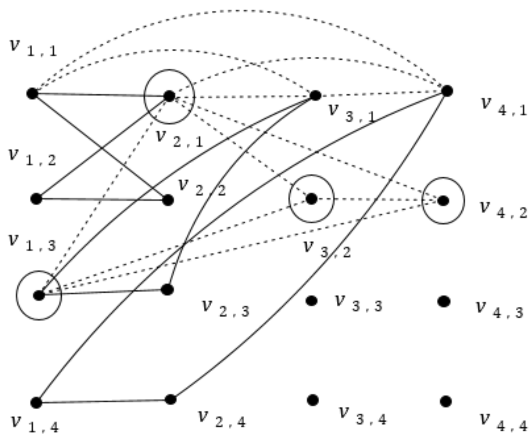

Figure 7:

For $\{v_{1,3},v_{2,3},v_{3,2},v_{4,1}\}$, $\{v_{1,3},v_{2,3},v_{3,3},v_{4,1}\}$ and $\{v_{1,3},v_{2,3},v_{3,4},v_{4,1}\}$ not to induce a blue K_4, $(v_{1,3},v_{2,3})$ has to be a red edge (since the red degrees of both $v_{1,3}$ and $v_{2,3}$ must be at most two). Similarly, in order for $\{v_{1,4},v_{2,4},v_{3,1},v_{4,2}\}$, $\{v_{1,4},v_{2,4},v_{3,1},v_{4,3}\}$ and $\{v_{1,4},v_{2,4},v_{3,1},v_{4,4}\}$ not to induce a blue K_4, the edge $(v_{1,4},v_{2,4})$ has to be red. (since the red degrees of both $v_{1,4}$ and $v_{2,4}$ must be at most two). In order to avoid a red $K_{1,3}$, $v_{4,2}$ cannot be adjacent to all three vertices of $\{v_{3,2},v_{3,3},v_{3,4}\}$ in red. Therefore without loss of generality, we may assume that $(v_{3,2},v_{4,2})$ has to be a blue edge. This gives rise to Figure 7. Then as indicated in Figure 7, $\{v_{1,3},v_{2,1},v_{3,2},v_{4,2}\}$ will induce a blue K_4, a contradiction.

Case III. $(v_{1,2},v_{3,1})$,$(v_{1,3},v_{3,1})$ and $(v_{1,4},v_{4,1})$ are red edges. The resulting graph is represented in Figure 8.

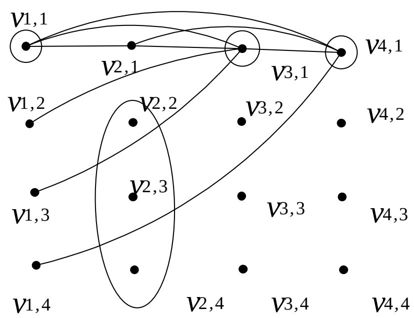

Figure 8:

It is evident from Figure 8 that in order to avoid a blue K_4 every single vertex in S_2 has to be adjacent to a vertex of S_3 in red. But this will force one of the three vertices of S_3 to have red degree greater than two. Thus, H_R will contain a red $K_{1,3}$, a contradiction. From the three cases it follows that, $m_4(K_{1,3},K_4) \leq 4$. That is, $m_4(K_{1,3},K_4) = 4$ as required.
Next let us consider the remaining case $j = 3$. Let t be an arbitrary integer. Consider the colouring of $K_{3\times t}$ generated by $H_B = K_{3\times t}$. Then, $K_{3\times t}$ has no red $K_{1,3}$ or a blue K_4. Hence, $m_3(K_{1,3},K_4) \geq t$ for any integer t. Therefore, we can conclude that $m_3(K_{1,3},K_4) = \infty$.

Theorem 2. *If $j \geq 3$, then*

$$m_j(K_{1,3},K_{1,3}+e)=\begin{cases} 1 & j\geq 7 \\ 2 & j\in\{5,6\} \\ 3 & j\in\{3,4\} \end{cases}$$

Proof: If $j \geq 7$, since $r(K_{1,3},K_{1,3}+e) = 7$ (see [2]), we get $m_j(K_{1,3},K_{1,3}+ e) = 1$.

Colour the graph $K_{6\times 1}$ such that $H_R = 2K_3$. Then the graph has no red $K_{1,3}$ and has no blue $K_{1,3}+e$. Therefore, $m_6(K_{1,3},K_{1,3}+e) \geq 2$. Next to show $m_5(K_{1,3},K_{1,3}+e) \leq 2$, consider any red and blue colouring of $K_{5\times 2}$, such that H_R contains no red $K_{1,3}$ and H_B contains no blue $K_{1,3}+e$. From [5], there is a blue C_3, in H_B as $m_5(K_{1,3},C_3) = 2$. Without loss of generality assume that the blue C_3, is induced by say $v_{1,1},v_{2,1},v_{3,1}$. But then if we consider the vertex $v_{1,1}$ it cannot be adjacent in blue to to any of the vertices of $v_{4,1},v_{4,2},v_{5,1}$ as it would result in a blue $K_{1,3}+e$. Therefore, $v_{1,1}$ will be a root of a red $K_{1,3}$, a contradiction. Thus, $2 \leq m_6(K_{1,3},K_{1,3}+e) \leq m_5(K_{1,3},K_{1,3}+e) \leq 2$. That is, $m_j(K_{1,3},K_{1,3}+e) =$ 3 for $j \in \{5,6\}$ as required.

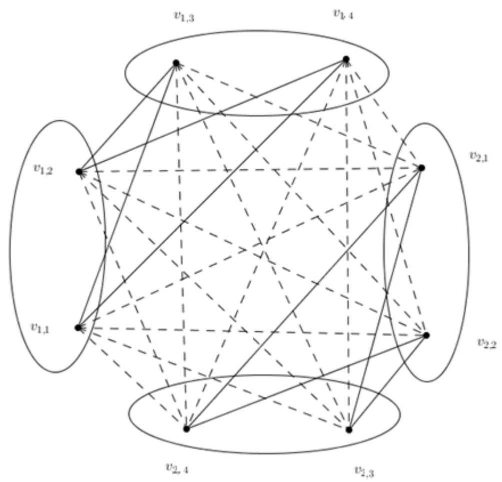

Figure 9: H_R and H_B graph related to the proof of $m_4(K_{1,3},K_{1,3}+e) \geq 3$

Consider the case $j \in \{3,4\}$. Colour the graph $K_{4\times 2}$, such that the red graph H_R equals to a $2C_4$ whereas, the blue graph H_B equals a $K_{4,4}$ as illustrated in Figure 9. Then the graph has no red $K_{1,3}$ and has no blue $K_{1,3}+e$. Therefore, $m_4(K_{1,3},K_{1,3}+e) \geq 3$.

To show, $m_3(K_{1,3},K_{1,3}+e) \leq 3$, consider any red and blue colouring of $K_{3\times 3}$ such that H_R contains no red $K_{1,3}$ and H_B contains no blue $K_{1,3}+e$. From [5], there is a blue C_3 in H_B as $m_3(K_{1,3},C_3) = 3$. Without loss of generality, assume that the blue C_3 is induced by say $v_{1,1},v_{2,1},v_{3,1}$. As H_B contains no blue $K_{1,3}+e$ we know that $(v_{3,1},v_{1,2})$,$(v_{3,1},v_{2,2})$ and $(v_{3,1},v_{1,3})$ must be red edges. However, this gives a red $K_{1,3}$ with $v_{3,1}$ as the root, a contradiction. Therefore, $m_3(K_{1,3},K_{1,3}+e) \leq 3$. That is, $m_j(K_{1,3},K_{1,3}+e) = 3$ for $j \in \{3,4\}$ as required. □

The theorem listed below corresponding to row 6 and row 8 is somewhat straight forward to prove (also can be proved using a Sage program) and therefore left for the reader to verify.

Theorem 3. *If* $j \geq 3$, *then*

$$m_j(K_{1,3},K_{1,3}) = m_j(K_{1,3},C_4) = \begin{cases} 1 & j \geq 6 \\ 2 & j = \{4,5\} \\ 3 & j = 3 \end{cases}$$

4. Size Ramsey numbers $m_j(K_{1,3},G)$ when G is disconnected graph on 4 vertices

We have already dealt with all cases excluding $G = 2K_2$. We will deal with this in the following theorem.

Theorem 4. *If* $j \geq 3$, *then*

$$m_j(K_{1,3},2K_2) = \begin{cases} 2 & \text{if } j \in \{3,4\} \\ 1 & \text{if } j \geq 5 \end{cases}$$

Proof: Clearly $m_j(K_{1,3}, 2K_2) = 1$ when $j \geq 5$, as $r(K_{1,3}, 2K_2) = 5$ (see [2]).
When $j \in \{3,4\}$, consider the colouring of $K_{4\times1}$ generated by $H_R = C_3$. Then, $K_{4\times1}$ has no red $K_{1,3}$ or a blue $2K_2$. Therefore, we obtain that $m_2(K_{1,3},2K_2) \geq 2$. That is, $m_2(K_{1,3},2K_2) = 2$.

To show $m_3(K_{1,3},2K_2) \leq 2$, consider any red and blue colouring of $K_{3\times2}$, such that H_R contains no red $K_{1,3}$ and H_B contains no blue $2K_2$. Since H_R contains no red $K_{1,3}$ we get $\delta(H_B) \geq 2$. As $\delta(H_B) \geq 2$, we may assume that $v_{1,1}$, will have two neighbours, denoted by x and y such that $(v_{1,1},x)$ and $(v_{1,1},y)$ are blue edges. Then as $v_{1,2}$ also has two blue neighbours, this will result in two blue independent edges with one edge adjacent in blue to $v_{1,2}$ and the other adjacent in blue to $v_{1,2}$. That is, we get a blue $2K_2$, a contradiction. That is, $m_3(K_{1,3},2K_2) \leq 2$. Therefore, $m_3(K_{1,3},2K_2) = 2$. □

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