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A New Numerical Approach for Solving Initial Value Problems of Ordinary Differential Equations

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Abstract. In this article, we developed a new numerical approach which is mainly concentrates to solve some complicated initial value problems of ordinary differential equations. The complete breakdown of this new approach derivation is presented here. In our future work, we will examine on the main properties of the technique namely consistency, convergence and feasibility. The expansion of this new numerical scheme shall be worked-on and comparison is also be made with some existing methods.

***Keywords*:** Numerical approach, ordinary differential equations, numerical scheme expansion

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1. Introduction

In mathematical modeling differential equations are generally used in the field of science and engineering. In the field of mathematical physics different problems are arise as the form of differential equations. These types of differential equations may the formation either ordinary differential equations or partial differential equations. Practically, most of the models of the problem which are formulated by means of these equations are so complicated to determine the exact solution and one of two approaches is taken to approximate the solution. The first technique is to reduce the differential equation in to one that can be solved exactly and then use the result of the reduced equation to approximate the solution to the original problem. Another technique, which we will verify in this article, uses methods of approximations the solution of original problem. This is the technique that is generally taken as the approximation methods give more perfect results and relative error information. Numerical methods are usually used for solving mathematical problems that are articulated in the field of science and engineering in which case the determination of the exact solution is so hard or impossible. Only a few numbers of differential equations can be solved analytically. Consequently, to obtain the analytical solution for differential equation there exist different methods. A huge number of differential equations are unable to determine the solution in closed form using familiar analytical methods, in which case we apply numerical technique for solving a differential equation under certain initial restriction or restrictions. There exist different kinds of practical numerical methods for finding the solution of initial value problem of ordinary differential equations.

Many Numerical researchers namely Ogunrinde [1], Fatunla [2], Butcher [3], Liu, Turner [4] and even Ibijola [5] and so on, have established schemes in order to solve some initial value problems of ordinary differential equations. Enright, Fellen and Sedgwick [6] have developed a numerical method which compares the numerical solution of ordinary differential equations. Yuan and Agrawal [7], established a scheme for fractional derivatives, on the other hand Obaymi [8,9] also studied on some approximation techniques which were used to derive stable non-standard finite difference schemes. And Ibijola [10] focus on the convergence, consistency and stability of a method of integration of ordinary differential equations. [11-13] solved ODE with numerical examples. The efficiency of all their efforts made for stability, accuracy, convergence and consistency of the methods. The accuracy property of different methods can be considered an order and convergence as well as truncation error co-efficient.

In this paper, we have established a new numerical technique with some particular properties to determine the solution of initial value problems of ordinary differential equations based on the local representation of theoretical form.

Let us consider

$$y' = \frac{dy}{dx} = f(x,y), \qquad y(a) = y_0 \tag{1}$$

is interpolating by the function

$$F(x) = C_0 + C_1x^2 + C_2e^{x^2-1} + Dsin(x^2-1) \tag{2}$$

where C_0, C_1, C_2 and D are real undetermined co-efficients.

2. Derivation of the new approach

Let us suppose that y_k is a numerical estimation of the theoretical solution $y(x)$ and $f_k = f(x_k,\ y_k)$.

Now we define the mesh points as follows: $x_k = a + kh$; *k=0,1,2,3,… …….*

We proceed to derive the new technique is as follows:

$$F'(x) = 2C_1x + 2C_2xe^{x^2-1} + 2Dx\cos(x^2-1) \tag{3}$$

$$F''(x) = 2C_1 + 2C_2\left(2x^2e^{x^2-1} + e^{x^2-1}\right) - 2D\{2x^2sin(x^2-1) - \cos(x^2-1)\} \tag{4}$$

Similarly,

$$F'''(x) = 2C_2\left(4x^3e^{x^2-1} + 6xe^{x^2-1}\right) - 2D\{4x^3\cos(x^2-1) + 6xsin(x^2-1)\} \tag{5}$$

$F^{iv}(x) = 2C_2\left(8x^4e^{x^2-1} + 24x^2e^{x^2-1} + 6e^{x^2-1}\right) - 2D\{-8x^4 sin(x^2-1) + 24x^2\cos(x^2-1) + 6\, sin(x^2-1)\}$ (6)

From equation (2), we have,

$$F(x) = C_0 + C_1x + C_2e^{x^2-1} + Dsin(x^2-1)$$

Or, $C_0 = F(x) - C_1x - C_2e^{x^2-1} - Dsin(x^2-1)$ (7)

And from equation (3), we get,

$C_1 = \frac{F'(x)}{2x} - C_2e^{x^2-1} - D\cos(x^2-1)$ (8)

From equation (4), we get,

$C_2 = \frac{F''(x) - 2C_1 + 2D\{2x^2 sin(x^2-1) - \cos(x^2-1)\}}{2\left(2x^2e^{x^2-1} + e^{x^2-1}\right)}$ (9)

Also from equation (5), we get,

$D = \frac{F'''(x) - 2C_2\left(4x^3e^{x^2-1} + 6xe^{x^2-1}\right)}{-2\{4x^3\cos(x^2-1) + 6xsin(x^2-1)\}}$ (10)

Now substituting the value of C_1 from (8) into (9), we get,

$C_2 = \frac{F''(x) - F'(x) + 2D\{x\cos(x^2-1) - \cos(x^2-1)\} + 4Dx^2 sin(x^2-1)}{4x^3e^{x^2-1}}$ (11)

Putting the value of C_2 from (11) into (10), we get,

$D = \frac{x^4F'''(x) + (-2x^4 - 3x^2)F''(x) + (2x^4 + 3x^2)F'(x)}{\{-8x^7 + 2x^5 - 2x^4\}\cos(x^2-1) + \{-12x^5 + 8x^8 + 3\}\sin(x^2-1)}$ (12)

By substituting the value of D from (12) into (11), we get,

$$C_2 = \frac{\begin{matrix}[(6x^2 - 6x^3 - 2x^4 + 2x^5 + 8x^7)\cos(x^2-1) + (-12x^4 + 12x^5 + 8x^6 - 8x^8 - 3)\sin(x^2-1)]F'(x) \\ +[\{6x^3 - 4x^4 - 2x^5 - 8x^7\}\cos(x^2-1) + \{-24x^5 - 8x^6 + 8x^8 + 3\}\sin(x^2-1)]F''(x) \\ +\{(2x^5 - 2x^4)\cos(x^2-1) + 4x^6 sin(x^2-1)\}F'''(x)\end{matrix}}{[\{-8x^7 + 2x^5 - 2x^4\}\cos(x^2-1) + \{-12x^5 + 8x^8 + 3\}\sin(x^2-1)]4x^3e^{x^2-1}} \quad (13)$$

Putting the value of D from (11) and C_2 from (12) into (8), we get,

$$C_1 = \frac{1}{2x}F'(x) - C_2e^{x^2-1} - D\cos(x^2-1) = \frac{1}{2x}F'(x)$$

$$-\left\{\frac{\begin{matrix}[(6x^2 - 6x^3 - 2x^4 + 2x^5 + 8x^7)\cos(x^2-1) + (-12x^4 + 12x^5 + 8x^6 - 8x^8 - 3)\sin(x^2-1)]F'(x) \\ +[\{6x^3 - 4x^4 - 2x^5 - 8x^7\}\cos(x^2-1) + \{-24x^5 - 8x^6 + 8x^8 + 3\}\sin(x^2-1)]F''(x) \\ +\{(2x^5 - 2x^4)\cos(x^2-1) + 4x^6 sin(x^2-1)\}F'''(x)\end{matrix}}{[\{-8x^7 + 2x^5 - 2x^4\}\cos(x^2-1) + \{-12x^5 + 8x^8 + 3\}\sin(x^2-1)]4x^3}\right\}$$

$$-\cos(x^2-1)\left\{\frac{x^4F'''(x) + (-2x^4 - 3x^2)F''(x) + (2x^4 + 3x^2)F'(x)}{\{-8x^7 + 2x^5 - 2x^4\}\cos(x^2-1) + \{-12x^5 + 8x^8 + 3\}\sin(x^2-1)}\right\} \quad (14)$$

Let us consider,

$$P = \left\{\frac{\begin{matrix}[(6x^2 - 6x^3 - 2x^4 + 2x^5 + 8x^7)\cos(x^2-1) + (-12x^4 + 12x^5 + 8x^6 - 8x^8 - 3)\sin(x^2-1)]F'(x) \\ +[\{6x^3 - 4x^4 - 2x^5 - 8x^7\}\cos(x^2-1) + \{-24x^5 - 8x^6 + 8x^8 + 3\}\sin(x^2-1)]F''(x) \\ +\{(2x^5 - 2x^4)\cos(x^2-1) + 4x^6 sin(x^2-1)\}F'''(x)\end{matrix}}{[\{-8x^7 + 2x^5 - 2x^4\}\cos(x^2-1) + \{-12x^5 + 8x^8 + 3\}\sin(x^2-1)]4x^3}\right\}$$

And $Q = \cos(x^2 - 1)\left\{\frac{x^4F'''(x)+(-2x^4-3x^2)F''(x)+(2x^4+3x^2)F'(x)}{\{-8x^7+2x^5-2x^4\}\cos(x^2-1)+\{-12x^5+8x^8+3\}\sin(x^2-1)}\right\}$

Hence the equation (14) becomes,

$$C_1 = \frac{1}{2x}F'(x) - P - Q \quad (15)$$

Now applying the following restrictions on the interpolating function (2) in the following way:

Condition 1. The interpolating function (2) must be coincide with the theoretical solution at$x = x_k$ and $x = x_{k+1}$, such that

$$F(x_k) = C_0 + C_1x_k^2 + C_2e^{x_k^2-1} + Dsin(x_k^2 - 1)$$

And $F(x_{k+1}) = C_0 + C_1x_{k+1}^2 + C_2e^{x_{k+1}^2-1} + Dsin(x_{k+1}^2 - 1)$

Condition 2. The derivatives $F'(x), F''(x), F'''(x)$ and $F^k(x)$ are coincide with $f(x), f'(x), f''(x)$ and $f^{k-1}(x)$ respectively, that is,

$$F'(x) = f_k$$
$$F''(x) = f_k'$$
$$F'''(x) = f_k''$$
$$F^{iv}(x) = f_k'''$$

From the above conditions (1) and (2), it follows that, if $F(x_{k+1}) - F(x_k) = y_{k+1} - y_k$

Then we have,

$$C_0 + C_1x_{k+1}^2 + C_2e^{x_{k+1}^2-1} + Dsin\left(x_{k+1}^2 - 1\right) - \{C_0 + C_1x_k^2 + C_2e^{x_k^2-1} + Dsin(x_k^2 - 1)\} = y_{k+1} - y_k$$

so, $y_{k+1} = y_k + C_1\left(x_{k+1}^2 - x_k^2\right) + C_2\left(e^{x_{k+1}^2-1} - e^{x_k^2-1}\right) + D\left\{sin\left(x_{k+1}^2 - 1\right) - sin(x_k^2 - 1)\right\}$ (16)

Let us assume that

$x_k = a + kh$ then $x_k^2 = (a + kh)^2 = a^2 + 2akh + (kh)^2$ (17)

Also $x_{k+1} = a + (k + 1)h$ so $x_{k+1}^2 = \{a + (k + 1)h\}^2$

hence, $x_{k+1}^2 = a^2 + 2akh + 2ah + (kh)^2 + 2kh^2 + h^2$ (18)

Now we calculate, $x_{k+1}^2 - x_k^2 = 2h(a + kh) + h^2$ (19)

Similarly,

$$e^{x_{k+1}^2-1} - e^{x_k^2-1} = e^{a^2+2akh+2ah+(kh)^2+2kh^2+h^2-1} - e^{a^2+2akh+(kh)^2-1}$$

$$e^{x_{k+1}^2-1} - e^{x_k^2-1} = e^{a^2+2akh+(kh)^2-1}\left[e^{2h(a+kh)+h^2} - 1\right] \quad (20)$$

And

$sin\left(x_{k+1}^2 - 1\right) - sin\left(x_k^2 - 1\right) = sin\{a^2 + 2akh + 2ah + (kh)^2 + 2kh^2 + h^2 - 1\} - \sin\{a^2 + 2akh + (kh)^2 - 1\}$

(21)

Putting (17) through (21) into equation (16) then we obtain our required numerical approach

$$
\begin{aligned}
y_{k+1} &= \\
y_k &+ \left[\frac{1}{2x_k}F'(x_k) - P - Q\right] \times \{2h(a+kh)+h^2\} + \\
&\left[\frac{\begin{array}{c}[(6x^2-6x^3-2x^4+2x^5+8x^7)\cos(x^2-1)+(-12x^4+12x^5+8x^6-8x^8-3)\sin(x^2-1)]F'(x) \\ +[\{6x^3-4x^4-2x^5-8x^7\}\cos(x^2-1)+\{-24x^5-8x^6+8x^8+3\}\sin(x^2-1)]F''(x) \\ +\{(2x^5-2x^4)\cos(x^2-1)+4x^6 sin(x^2-1)\}F'''(x)\end{array}}{[\{-8x^7+2x^5-2x^4\}\cos(x^2-1)+\{-12x^5+8x^8+3\}\sin(x^2-1)]4x^3e^{x^2-1}}\right] \times \\
&e^{\{a^2+2akh+(kh)^2-1\}}\left\{e^{2ah+2kh^2+h^2}-1\right\} + \\
&\left[\frac{x^4F'''(x)+(-2x^4-3x^2)F''(x)+(2x^4+3x^2)F'(x)}{\{-8x^7+2x^5-2x^4\}\cos(x^2-1)+\{-12x^5+8x^8+3\}\sin(x^2-1)}\right] \times [\sin\{(a^2+2akh+2ah+(kh)^2+ \\
&2kh^2+h^2)-1\} - \sin\{(a^2+2akh+(kh)^2)-1\}] \qquad (22)
\end{aligned}
$$

Equation (22) is the New Numerical Approach for the solution of ordinary differential equations with the given initial values.

3. Conclusion

Our main objective is to establish a new approach as a recommendation whose numerical approximation result could be coincides with some existing method of solution of different initial value problems of ordinary differential equations. So this paper has been capable to initiate the new approach as a proposal. In our future research, we will pay more concentration to validate this approach with some numerical examples. We will check some basic characteristics such as the accuracy, consistency, reliability, stability of the proposed new numerical approach by solving (1) and then discuss the relative error, truncation error on the comparison with some existing standard methods.

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