Annals of Pure and Applied Mathematics
Vol. 17, No. 2, 2018, 193-195
ISSN: 2279-087X (P), 2279-0888(online)
Published on 2 Jun 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v17n2a4

Annals of Pure and Applied Mathematics

The Class of Diophantine Equations $p^4 + q^y = z^4$ when $y = 1, 2, 3$ is Insolvable for all Primes p and q

Nechemia Burshtein

117 Arlozorov Street, Tel Aviv 6209814, Israel
Email: anb17@netvision.net.il

Received 8 April 2018; accepted 29 May 2018

Abstract. In this short paper, we investigate the equation $p^4 + q^y = z^4$ when p, q are primes and $y = 1, 2, 3$. In a very simple manner, we show that each of the three equations has no solutions. When q is composite, or when both p and q are composites, some solutions are also exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 5, 7].

In 1637, Fermat (1601 – 1665) stated that the Diophantine equation $x^n + y^n = z^n$, with integral $n > 2$, has no solutions in positive integers x, y, z. This is known as Fermat's "Last Theorem". In 1995, 358 years later, the validity of the Theorem was established and published by A. Wiles. Thus, for integral $n \geq 3$, the equation $p^n + q^n = z^n$ has no solutions in positive integers p, q, z.

One may now ask the question whether or not the equation $p^n + q^y = z^n$ has solutions for all values y where $1 \leq y \leq n - 1$. When $n = 3$, the author [3] established that the equation $p^3 + q^2 = z^3$ ($y = 2$) has exactly four solutions in all of which $p = 7$. In one solution q is prime, whereas in the other three solutions q is composite. For $n = 3$, the author [1] also considered the equation $p^3 + q^1 = z^3$ ($y = 1$) with primes p and q, and showed that the equation has infinitely many solutions.

In Section 2 of this short paper, we investigate the equation $p^n + q^y = z^n$ when $n = 4$ i.e., $p^4 + q^y = z^4$ for all values $y < 4$ when p and q are primes.

2. The equation $p^4 + q^y = z^4$ is insolvable for primes p, q and $y = 1, 2, 3$

In the following Theorem 2.1 when $n = 4$, with $y = 1, 2, 3$, and p, q are primes, the three equations $p^4 + q^y = z^4$ are considered. In a very simple and elementary way, it is shown that each of these equations has no solutions.

Theorem 2.1. Suppose that p, q are any two distinct primes. For each value $y = 1$, 2, 3, the respective equation $p^4 + q^y = z^4$ has no solutions.

Proof: We have the set of three equations

(a) $p^4 + q^1 = z^4$,

(b) $p^4 + q^2 = z^4$,

(c) $p^4 + q^3 = z^4$.

Each case will be considered separately.

The equation $p^4 + q^y = z^4$ yields

$$q^y = z^4 - p^4 = (z^2 - p^2)(z^2 + p^2) = (z - p)(z + p)(z^2 + p^2), \tag{1}$$

where q^y is the product of three distinct factors.

Suppose **(a)**, i.e., $p^4 + q^1 = z^4$.

Hence by (1)

$$q = (z - p)(z + p)(z^2 + p^2). \tag{2}$$

Since q is prime, and has the only two divisors 1 and q, it clearly follows that equation (2) is therefore impossible.

Thus $p^4 + q^1 = z^4$ has no solutions for all primes p and q.

Suppose **(b)**, i.e., $p^4 + q^2 = z^4$. By (1) we have

$$q^2 = (z - p)(z + p)(z^2 + p^2). \tag{3}$$

The three divisors of q^2 are 1, q and q^2. It is easily seen that none of the three factors in (3) can be equal to either q or to q^2. Therefore equation (3) is impossible.

Hence $p^4 + q^2 = z^4$ has no solutions for all primes p and q.

Suppose **(c)**, i.e., $p^4 + q^3 = z^4$. From (1) we obtain

$$q^3 = (z - p)(z + p)(z^2 + p^2), \tag{4}$$

and the divisors of q^3 are 1, q, q^2 and q^3. Evidently, none of these divisors can be applied in any way to equation (4). It follows that equation (4) is impossible. Therefore, the equation $p^4 + q^3 = z^4$ has no solutions for all primes p and q.

This concludes the proof of Theorem 2.1. □

Final Remark. Suppose that the conditions in $p^4 + q^y = z^4$ are relaxed. For instance, p is prime, but q is composite. Then, when $y = 1$ and $z = p + 1$, the equation has infinitely many solutions. The first four such solutions are as follows:

$$(p,\ q,\ y = 1,\ z = p + 1) = (2, 65, 1, 3),\ (3, 175, 1, 4),\ (5, 671, 1, 6),\ (7, 1695, 1, 8).$$

Furthermore, when p, q are two composites, i.e., $p = C_1$, $q = C_2$, with $y = 1$ and $z = C_1 + 1$, we have the solution

$$(C_1,\ C_2,\ y = 1,\ z = C_1 + 1) = (4, 369, 1, 5).$$

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