Some Properties of 0-distributive Meet Semilattices

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Abstract. J.C. Varlet introduced the concept of 0-distributive lattices to generalize the notion of pseudo complemented lattices. A lattice L with 0 is called a 0-distributive lattice if for all \( a, b, c \in L \), \( a \land b = 0 = a \land c \) imply \( a \land (b \lor c) = 0 \). Of course every distributive lattice with 0 is 0-distributive. Also every pseudo complemented lattice is 0-distributive. Recently, Chakraborty and Talukder extended this concept for directed above meet semi lattices. A meet semi lattice S is called directed above if for all \( a, b \in S \), there exists \( c \in S \) such that \( c \geq a, b \). Again Y. Rav has extended the concept of 0-distributivity by introducing the notion of semi prime ideals in a lattice. Recently, Noor and Begum have studied the semi prime ideals in a directed above meet semi lattice. In this paper we have included several characterizations and properties of 0-distributive meet semi lattices. We proved that for a meet sub semi lattice A of S, \( A^0 = \{ x \in S : x \land a = 0 \text{ for some } a \in A \} \) is a semi prime ideal of S if and only if S is 0-distributive. Using different equivalent conditions of 0-distributive meet semi lattices we have given a ‘Separation theorem’ for \( \alpha \) - ideals.

Keywords: 0-distributive meet semi lattice, Semi prime ideal, Prime ideal, Maximal ideal, \( \alpha \) -ideal.

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1.Introduction

J.C. Varlet [7] first introduced the concept of 0-distributive lattices. Then many authors including [1,2,5] studied them for lattices and semilattices. By [2], a meet semilattice S with 0 is called 0-distributive if for all \( a, b, c \in S \) with \( a \land b = 0 = a \land c \) imply \( a \land d = 0 \) for some \( d \geq b, c \). We also know that a
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0-distributive meet semilattice is directed above. A meet semilattice $S$ is called *directed above* if for all $a, b \in S$, there exists $c \in S$ such that $c \geq a, b$.

A non-empty subset $I$ of a directed above meet semilattice $S$ is called a down set if for $x \in I$ and $y \leq x \ (y \in S)$ imply $y \in I$. Down set $I$ is called an ideal if for $x, y \in I$, there exists $z \geq x, y$ such that $z \in I$.

A non-empty subset $F$ of $S$ is called an upset if $x \in F$ and $y \geq x \ (y \in S)$ imply $y \in F$. An upset $F$ of $S$ is called a filter if for all $x, y \in F$, $x \land y \in F$. An ideal (down set) $P$ is called a prime ideal (down set) if $a \land b \in P$ implies either $a \in P$ or $b \in P$. A filter $Q$ of $S$ is called prime if $S - Q$ is a prime ideal.

A filter $F$ of $S$ is called a maximal filter if $F \neq S$ and it is not contained by any other proper filter of $S$. A prime down set $P$ is called a minimal prime down set if it does not contain any other prime down set of $S$.

Following Lemmas in lattices are due to [1] and [5], and also hold for meet semilattices by [2].

**Lemma 1.** A proper subset $F$ of a meet semilattice $S$ is maximal if and only if $S - F$ is a minimal prime down set. 

**Lemma 2.** Let $F$ be a proper filter of a meet semilattice $S$ with 0. Then there exists a maximal filter containing $F$.

Following result is due to [4,Lemma 5]

**Lemma 3.** Let $F$ be a filter and $I$ be an ideal of a directed above meet semilattice $S$, such that $F \cap I = \emptyset$. Then $F$ is a maximal filter disjoint from $I$ if and only if for each $a \not\in F$, there exists $b \in F$ such that $a \land b \in I$.

Let $S$ be a meet semilattice with 0. For a non-empty subset $A$ of $S$, we define $A^\perp = \{x \in S \mid x \land a = 0 \text{ for all } a \in A\}$. This is clearly a down set, but we can not prove that this is an ideal even in a distributive meet semilattice. If $L$ is a lattice with 0, then it is well known that $L$ is 0-distributive if and only if $I(L)$ is 0-distributive. Unfortunately, we can not prove or disprove that when $S$ is a 0-distributive meet semilattice, then $I(S)$ is 0-distributive. But if $I(S)$ is 0-distributive, then it is easy to prove that $S$ is also 0-distributive. We define $A^0 = \{x \in S \mid x \land a = 0 \text{ for some } a \in A\}$. This is obviously a down set. Moreover, $A \subseteq B$ implies $A^0 \subseteq B^0$. For any $a \in S$, it easy to check that $(a)^\perp = (a)^0 = [a]^0$.

Following result is due to [2].
Theorem 4. Let $S$ be a directed above meet semilattice with 0. Then the following conditions are equivalent.

(i) $S$ is 0-distributive

(ii) For each $a \in S$, $(a) \downarrow = (a)^0 = [a]^0$ is an ideal.

(iii) Every maximal filter of $S$ is prime. \(\square\)

Since in a 0-distributive meet semilattice $S$, for each $a \in S$, $(a) \downarrow$ is an ideal, so we prefer to denote it by $(a)^*$. Y Rav [6] have generalized the concept of 0-distributive lattices and introduced the notion of semi prime ideals in lattices. In a very recent paper [4] have extended the concept in a directed above meet semilattice. In a directed above meet semilattice $S$, an ideal $J$ is called a semi prime ideal if for all $x, y, z \in S$, $x \wedge y \in J$, $x \wedge z \in J$ imply $x \wedge d \in J$ for some $d \geq y, z$. In a distributive semilattice, every ideal is semi prime. Moreover, the semilattice itself is obviously a semi prime ideal. Also, every prime ideal of $S$ is semi prime.

Theorem 5. For any meet sub semilattice $A$ of a directed above meet semi lattice $S$ with 0, $A^0$ is a semi prime ideal of $S$ if and only if $S$ is 0-distributive.

Proof: Suppose $S$ is 0-distributive. We already know that $A^0$ is a down set, Now let $x, y \in A^0$. Then $x \wedge a = 0 = y \wedge b$ for some $a, b \in A$. Then $x \wedge a \wedge b = 0 = y \wedge a \wedge b$. Since $S$ is 0-distributive, so $(a \wedge b) \wedge d = 0$ for some $d \geq x, y$. Now $a \wedge b \in A$ implies $d \in A^0$, and so $A^0$ is an ideal. Finally let $x \wedge y \in A^0$, and $x \wedge z \in A^0$. Then $x \wedge y \wedge a_i = 0 = x \wedge z \wedge b_i$ for some $a_i, b_i \in A$. Thus $x \wedge a_i \wedge b_i \wedge y = 0 = x \wedge a_i \wedge b_i \wedge z$. Then by the 0-distributive property, $x \wedge a_i \wedge b_i \wedge d_1 = 0$ for some $d_1 \geq y, z$. Thus $x \wedge d_1 \in A^0$ as $a_i \wedge b_i \in A$. Therefore $A^0$ is semi prime. Conversely, if $A^0$ is a semi prime ideal for every meet sub semilattice $A$ of $S$, then in particular $(a)^0$ is an ideal. Thus $S$ is 0-distributive by Theorem 4. \(\square\)

Following characterization of semi prime ideals is due to [4].

Theorem 6. Let $S$ be a directed above meet Semilattice with 0 and $J$ be an ideal of $S$.

Then the following conditions are equivalent.

(i) $J$ is semi prime

(ii) Every maximal filter disjoint to $J$ is prime. \(\square\)

Thus we have the following separation theorem.
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**Theorem 7.** Let $S$ be a 0-distributive meet semilattice and $A$ be a meet subsemilattice of $S$. Then for a filter $F$ disjoint from $A^0$, there exists a prime ideal containing $A^0$ and disjoint from $F$. □

**Lemma 8.** Let $A$ and $B$ be filters of a directed above meet semilattice $S$ with 0, such that $A \cap B^0 = \varphi$. Then there exists a minimal prime down set containing $B^0$ and disjoint from $A$.

**Proof:** Observe that $0 \not\in A \lor B$. For if $0 \in A \lor B$, then $0 \geq a \land b$ for some $a \in A$, $b \in B$. That is, $a \land b = 0$, which implies $a \in B^0$ gives a contradiction. It follows that $A \lor B$ is a proper filter of $S$. Then by Lemma 2, $A \lor B \subseteq M$ for some maximal filter $M$. If $x \in M \cap B^0$, then $x \in M$ and $x \land b_i = 0$ for some $b_i \in B \subseteq M$. This implies $0 \in M$ which is a contradiction as $M$ is maximal. Thus, $M \cap B^0 = \varphi$. Then by Lemma 1, $S - M$ is a minimal prime down set containing $B^0$. Moreover, $(S - M) \cap A = \varphi$. □

**Lemma 9.** Let $A$ be a filter of a directed above meet semilattice $S$ with 0. Then $A^0$ is the intersection of all the minimal prime down sets disjoint from $A$.

**Proof:** Let $N$ be any minimal prime down set disjoint from $A$. If $x \in A^0$, then $x \land a = 0$ for some $a \in A$ and so $x \in N$ as $N$ is prime.

Conversely, let $y \in S - A^0$. Then $y \land a \neq 0$ for all $a \in A$. Hence $A \lor \{y\}$ is a proper filter of $S$. Then by Lemma 2, $A \lor \{y\} \subseteq M$ for some maximal filter $M$. Thus by Lemma 1, $S - M$ is a minimal prime down set. Clearly $(S - M) \cap A = \varphi$ and $y \notin S - M$. □

Now we include some characterization of 0-distributive meet semilattices.

**Theorem 10.** Let $S$ be a directed above meet semilattice with 0. Then the following statements are equivalent.

(i) $S$ is 0-distributive.

(ii) For each $a \in S$, $(a)^0$ is a semi prime ideal.

(iii) For any three filters $A$, $B$, $C$ of $S$,

$(A \lor (B \land C))^0 = (A \lor B)^0 \land (A \lor C)^0$

(iv) For all $a, b, c \in S$,

$((a) \lor ((b) \land [c]))^0 = ((a) \lor [b])^0 \land ([a] \lor [c])^0$
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(v) For all \(a, b, c \in S\), \((a \land d)^0 = (a \land b)^0 \cap (a \land c)^0\) for some \(d \geq b, c\).

Proof: (i) \(\iff\) (ii). Follows by theorem 4.
(i) \(\Rightarrow\) (iii). Let \(x \in (A \lor B)^0 \cap (A \lor C)^0\). Then \(x \in (A \lor B)^0\) and \(x \in (A \lor C)^0\). Thus \(x \land f = x \land g\) for some \(f \in A \lor B\) and \(g \in A \lor C\). Then \(f \geq a_i \land b\) and \(g \geq a_j \land c\) for some \(a_i, a_j \in A\), \(b \in B\), \(c \in C\). This implies \(x \land a_i \land b = x \land a_j \land c\) and so \(x \land a_i \land a_j \land b = 0 = x \land a_i \land a_j \land c\). Since \(S\) is 0-distributive, so \(x \land a_i \land a_j \land d = 0\) for some \(d \geq b, c\). Now \(a_i \land a_j \in A\) and \(d \in B \cap C\). Therefore, \(((a_i \land a_j) \land d \in A \lor (B \cap C))\) and so \(x \in (A \lor (B \cap C))^0\). The reverse inclusion is trivial as \(A \lor (B \cap C) \subseteq A \lor B, A \lor C\). Hence (iii) holds.

(iii) \(\Rightarrow\) (iv) is trivial by considering \(A = [a], B = [b]\) and \(C = [c]\) in (iii).
(iv) \(\Rightarrow\) (v). Let (iv) holds. Suppose \(x \in (a \land b)^0 \cap (a \land c)^0\). Then by (iv) \(x \in ([a] \land [b])^0 \cap ([a] \land [c])^0\) and \([a] \land [b] \subseteq [a] \land [c]\). This implies \(x \land f = 0\) for some \(f \in [a] \lor ([b] \land [c])\). Then \(f \geq a \land d\) for some \(d \in [b] \land [c]\). That is, \(f \geq a \land d\) for some \(d \geq b, c\). It follows that \(x \land a \land d = 0\) and so \(x \in (a \land d)^0\). On the other hand, \([a] \lor [d] \subseteq [a] \lor [b]\) and \([a] \lor [d] \subseteq [a] \lor [c]\) implies that \((a \land d)^0 \subseteq (a \land b)^0 \cap (a \land c)^0\). Therefore (v) holds.

(v) \(\Rightarrow\) (i). Suppose (v) holds. Let \(a, b, c \in S\) with \(a \land b = 0 = a \land c\). Then \(a \land (a \land b) = 0 = a \land (a \land c)\) implies \(a \in (a \land b)^0 \cap (a \land c)^0 = (a \land d)^0\) for some \(d \geq b, c\). Thus, \(a \land (a \land d) = 0\) for some \(d \geq b, c\). That is \(a \land d = 0\) for some \(d \geq b, c\). and so \(S\) is 0–distributive.

Now we include few more characterizations of 0-distributive semi lattices.

**Theorem 11.** Let \(S\) be a directed above meet semi lattice with 0. Then the following are equivalent.

(i) \(S\) is 0-distributive.

(ii) For any three filters \(A, B, C\) of \(L\),
\[
((A \land B) \lor (A \land C))^0 = A^0 \land (B \lor C)^0
\]

(iii) For any two filters \(A, B\) of \(S\), \((A \lor B)^0 = A^0 \land B^0\)

(iv) For all \(a, b \in S\), \((a)^0 \land (b)^0 = (d)^0\) for some \(d \geq b, c\).

(v) For all \(a, b \in S\), \((a)^* \land (b)^* = (d)^*\) for some \(d \geq b, c\).
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**Proof:** (i) $\Rightarrow$ (ii). Suppose $S$ is 0-distributive, Since $(A \cap B) \vee (A \cap C) \subseteq A$ and $B \vee C$, so $((A \cap B) \vee (A \cap C))^0 \subseteq A^0 \cap (B \vee C)^0$. Now suppose

$x \in A^0 \cap (B \vee C)^0$ Then $x \in A^0$ and $x \in (B \vee C)^0$. Thus $x \land a = 0$ for some $x \in A$ and $x \land d = 0$ for some $d \in B \vee C$. Now $d \in B \vee C$ implies $d \geq b \land c$ for some $b \in B, c \in C$. Hence $x \land a = 0 = x \land b \land c$. Then $x \land c \land a = 0 = x \land c \land b$. Since $S$ is 0-distributive, so $x \land c \land d_1 = 0$ for some $d_1 \geq a,b$. Then $d_1 \in A \cap B$. Now $x \land a = 0$ implies $x \land d_1 \land a = 0 = x \land d_1 \land c$. Again by the 0-distributivity, $x \land d_1 \land d_2 = 0$ for some $d_2 \geq a,c$ that is $d_2 \in A \cap C$. Therefore, $x \in ((A \cap B) \vee (A \cap C))^0$ and so (ii) holds.

(ii) $\Rightarrow$ (iii) is trivial by considering $B = C$ in (iii).

(iii) $\Rightarrow$ (iv). Choose $A = [a]$ and $B = [b]$ in (iii).

Now for all $d \geq a,b$, $[a] \supseteq [d]$ and $[b] \supseteq [d]$ and so $[d]^0 \subseteq (a)^0 \cap (b)^0$. Also by (iii), $(a)^0 \cap (b)^0 = ([a] \cap [b])^0$. Thus, $x \in (a)^0 \cap (b)^0$. implies $x \land d_1 = 0$ for some $d_1 \geq a,b$. That is, $x \in (d_1)^0$ for some $d_1 \geq a,b$. Thus (iv) holds.

(iv) $\Leftrightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i). Suppose (v) holds and for $a,b,c \in S$, $a \land b = 0 = a \land c$. Then $a \in \{b\}^* \land \{c\}^* = \{d\}^*$ for some $d \geq b,c$. Therefore, $a \land d = 0$ and so $S$ is 0-distributive. □

An ideal $I$ in a directed above meet semilattice $S$ with 0 is called an $\alpha$-ideal if for each $x \in I$, $\{x\}^0 \subseteq I$.

**Proposition 12.** If $I$ is an $\alpha$-ideal of a 0-distributive meet semilattice $S$, Then $I = \{y \in S | \{y\} \subseteq \{x\}^0 \text{ for some } x \in I\}$.

**Proof:** Let $y \in R.H.S$. Then $\{y\} \subseteq \{x\}^0 \subseteq I$. This implies $y \in I$. Conversely, let $y \in I$. Since $S$ is 0-distributive, so by theorem 4, $(y)^0$ is an ideal and $(y) \cap (y)^0 = (0)$. Thus, $(y) \subseteq (y)^0$, which implies $y \in R.H.S$. □

Prime separation theorem for $\alpha$-ideals in 0-distributive lattices was given in [3]. Now we include a prime separation theorem on $\alpha$-ideals for 0-distributive meet semilattices.

**Theorem 13.** Let $F$ be a filter and $I$ be an $\alpha$-ideal of a directed above meet semilattice $S$ with 0, such that $I \cap F = \emptyset$. If $I$ ($S$) is 0-distributive, then there exists a prime $\alpha$-ideal $P$ containing $I$ such that $P \cap F = \emptyset$. 

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Proof: By lemma 2, there exists a maximal filter $M$ containing $F$ and disjoint to $I$. Thus $P = S - M$ is a minimal prime down set containing $I$ and disjoint to $M$. Now let $p, q \in S - M$. Then by lemma 3, there exist $a, b \in M$ such that $a \land p \in I$ and $b \land q \in I$. Then by proposition 12, $(a \land p) \subseteq (x) \downarrow$ and $(b \land q) \subseteq (y) \downarrow$ for some $x, y \in I$. Thus $(a \land p) \cup (x) \downarrow = (b \land q) \cup (y) \downarrow = (0)$. This implies $(a \land b) \cup (x) \downarrow \cup (y) \downarrow \cup (p) = (0) = (a \land b) \cup (x) \downarrow \cup (y) \downarrow \cup (q)$.

Now as $I$ is an ideal, there exists $d \geq x, y$ such that $d \in I$. Again by Theorem 11 (v), $(x) \downarrow \cup (y) \downarrow = (d_2) \downarrow$ for some $d_2 \geq x, y$. Then $d = d_1 \land d_2 \in I$, and so $(d) \downarrow \subseteq (x) \downarrow \cup (y) \downarrow = (d_2) \downarrow \subseteq (d) \downarrow$. Thus $(x) \downarrow \cup (y) \downarrow = (d) \downarrow$ for some $d \in I, d \geq x, y$. Then we have $(a \land b) \cup (d) \downarrow \cup (p) = (0) = (a \land b) \cup (d) \downarrow \cup (q)$. Since $I(S)$ is 0-distributive, so $(a \land b) \cup (d) \downarrow \cup ((p) \cup (q)) = (0)$. Then $(a \land b) \cup (d) \downarrow \cup (t) = (0)$ for some $t \geq p, q$. Thus $(a \land b \land t) \subseteq (d) \downarrow \subseteq I \subseteq S - M$. But $a \land b \in M$ implies $t \in S - M$ as $S - M$ is prime. Therefore $P = S - M$ is an ideal. Now let $x \in P$. If $x \in I$, then $(x) \downarrow \subseteq I \subseteq P$ as $I$ is an $\alpha$-ideal. Finally if $x \in P - I$. Then again by Lemma 3, there exists $a \in M$ such that $a \land x \in I$. Thus $(a) \downarrow \cup (x) \downarrow \subseteq I \subseteq P$. Since $a \notin P$, so $(a) \downarrow \subseteq P$. Therefore, $(x) \downarrow \subseteq P$ as $P$ is prime, and hence $P$ is also an $\alpha$-ideal.

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