Ekman Boundary Layer Mixed Convective Heat Transfer Flow through a Porous Medium with Large Suction

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Abstract. An analytical investigation on a mixed convective heat transfer steady flow past a continuously moving semi-infinite vertical plate bounded by a porous medium with large suction is performed in a rotating system. The governing equations of the problem are transformed by usual similarity transformations. To solve the momentum and energy equations, the perturbation technique is used in this work. The shear stresses and Nusselt number are also calculated here. The obtained numerical values of velocities and temperature are plotted in figures. To observe the effects of various parameters on the above mentioned quantities, the results are discussed in detailed with the help of graphs as well as the tabulated values. Finally, a important conclusion is listed here.

Keywords: Ekman Boundary Layer, Mixed Convective, Heat Transfer, Suction

AMS Mathematics Subject Classification (2010): 58D30, 35Q35

1. Introduction
The flow through a porous medium plays a decisive role in many industrial applications. Porous media are very widely used to insulate a heated body to maintain its temperature. To make the heat insulation of the surface more effective, it is necessary to study the free convection flow through a porous medium. Raptis et. al. [1] have observed the steady free convective flow through a porous medium bounded by an infinite surface by use of the model of Yamamoto and Iwamura [2] for the flow near the surface. Three-dimensional free convective heat transfer flow through a porous medium has been studied by Chaudhury and Chand [3].
Greenspan [4] was the first author to recognize the Ekman layer situation and he observed that, in a rotating fluid near a flat plate an Ekman layer exists wherein the viscous and Coriolis forces are of the same order of magnitude. The steady and unsteady Ekman layers of an incompressible fluid have investigated as a basic boundary layers in a rotating fluid appearing in the oceanic, atmospheric, cosmic fluid dynamics and solar physics or geophysical problems. The Ekman layer flow on a horizontal plate has been studied by Batchelor [5]. Mazumder et. al. [6] have studied the flow and heat transfer in a hydro-magnetic Ekman layer on a porous plate with Hall effects. In a rotating system, a hydromagnetic free convective flow past an impulsively started vertical plate has been observed by Singh [7].

All the above problems are investigated for free convective flow of a fluid. However, the flow by mixed convection plays a special role in a number of industrial applications such as fiber and granular insulation, geothermal systems etc. Hence, our main aim is to investigate the Ekman boundary layer mixed convective heat transfer steady flow past a continuously moving semi-infinite vertical plate surrounded by a porous medium with large suction.

2. Mathematical Model of Flow
Consider a steady mixed convective heat transfer flow of an electrically conducting viscous fluid along an electrically non-conducting semi-infinite vertical plate embedded by a porous medium. The flow is also assumed to be in the \( x \)-direction which is taken along the plate in the upward direction and \( y \)-axis is normal to it. Initially, we consider that the plate as well as the fluid particles is at rest at the same temperature \( T(=T_\infty) \) where \( T_\infty \) denotes the uniform temperature. It is assumed that the plate be at rest after that the plate is to be moving with a constant velocity \( U_0 \) in its own plane. It is also considered that the system is allowed to rotate with a constant angular velocity \( \Omega \) about the \( y \)-axis. Hence the angular velocity vector is of the form \( \Omega = (0, -\Omega, 0) \).

In accordance with the usual Boussinesq’s approximation, the equations relevant to the present problem are governed by the following system of coupled non-linear partial differential equations under the Ekman boundary layer Phenomena,

Continuity equation,
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

Momentum equations,
\[
u \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} + 2 \Omega w - \frac{u}{K} u \right) \]
\[
u \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \nu \frac{\partial^2 w}{\partial y^2} - 2 u \Omega - \frac{v}{K} w \right)
\]
Ekman Boundary Layer Mixed Convective Heat Transfer Flow …..

Energy equation, \( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{v}{c_p} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \)

with the appropriate boundary conditions,
\( u = U_o, \quad v = V(x), \quad w = 0, \quad T = T_w \) at \( y \to 0 \)
\( u = 0, \quad v = 0, \quad w = 0, \quad T = T_w \) as \( y \to \infty \)

where \( y \) is the cartesian coordinate; \( u, v, w \) are velocity components; \( g \) is the local acceleration due to gravity, \( \beta \) is the thermal expansion coefficient, \( \nu \) is the kinematic viscosity, \( \rho \) is the density, \( \kappa \) is the thermal conductivity and \( c_p \) is the specific heat at constant pressure.

3. Mathematical Formulation
In order to obtain the similar solutions of the mathematical model of flow, it is required to introduce the following similarity variables,
\( \eta = y \sqrt{\frac{U_o}{2v}}, \quad \psi = \sqrt{2\nu_x U_o f^2(\eta)} \) and \( \theta(\eta) = \frac{T - T_w}{T_w - T_w} \)

Introducing the above stated variables, we have the followings,
\( u = U_o f'(\eta), \quad v = \sqrt{\frac{\theta}{2x}} [\eta f'(\eta) - f(\eta)] \) and \( w = U_o \theta(\eta) \)

Using the above relations, we obtain the dimensionless equations,
\( f''' + f f' + G_r \theta - \kappa f' + 2Eg = 0 \)
\( g'' + f g - \nu g - 2E f'' = 0 \)
\( \theta' + P_r f \theta' + P_e (f'' + g'') = 0 \)

where, \( E = \frac{2\Omega \nu}{U_o} \) (Ekman Number), \( P_r = \frac{\nu c_p}{\kappa} \) (Prandtl Number),
\( G_r = \frac{2x}{U_o^2} g \beta (T_w - T_w) \) (Grashof Number), \( E_c = \frac{U_o^2}{c_p (T_w - T_w)} \) (Eckert Number),
\( K = \frac{2\Omega \nu}{\kappa U_o} \) (Permeability Parameter) and \( f'(\eta), \ g(\eta), \ \theta(\eta) \) represent the non-dimensional primary velocity, secondary velocity and temperature respectively.

Also the boundary conditions are transformed to,
\( f = f_w, \quad f' = 1, \quad g = 0, \quad \theta = 1 \) at \( \eta = 0 \)
\( f' = 0, \quad g = 0, \quad \theta = 0 \) as \( \eta \to \infty \)

where, \( f_w = -V(x) \sqrt{\frac{2x}{U_o^2}} \) is the transpiration parameter. Here \( f_w < 0 \) indicates the suction and \( f_w > 0 \) denotes the injection.
4. Solution of the Problem

Since the solution is sought for the large suction, hence we make the following transformations,

$$\xi = \eta f(\eta), \quad f(\eta) = f,F(\xi), \quad g(\eta) = f^{-1}G(\xi) \quad \text{and} \quad \theta(\eta) = f^{-1}H(\xi).$$

Using the above quantities we have the following system of equations,

$$F'' + FF' = \varepsilon(\kappa F' - G, H - 2EG)$$
$$G'' + FG' = \varepsilon (\kappa G + 2EF')$$

$$H'' + P_1 F'_1 H' + P_2 E_1 \frac{1}{\varepsilon} (F'' + G'^2)$$

with the boundary conditions,

$$F = 1, \quad F' = e, \quad G = 0, \quad H = e \quad \text{at} \quad ? = 0$$
$$F' = 0, \quad G = 0, \quad H = 0 \quad \text{as} \quad ? \to \infty$$

where, $\varepsilon = \frac{1}{f_w^2}$ is very small as for the large suction ($f_w > 1$). Therefore we can expand $F$, $G$ and $H$ in terms of the small perturbation quantity $e$ as follows,

$$F(\xi) = 1 + \varepsilon F_1(\xi) + \varepsilon^2 F_2(\xi) + \varepsilon^3 F_3(\xi) + \ldots \ldots$$

$$G(\xi) = \varepsilon G_1(\xi) + \varepsilon^2 G_2(\xi) + \varepsilon^3 G_3(\xi) + \ldots \ldots$$

$$H(\xi) = \varepsilon H_1(\xi) + \varepsilon^2 H_2(\xi) + \varepsilon^3 H_3(\xi) + \ldots \ldots$$

Introducing $F(\xi)$, $G(\xi)$ and $H(\xi)$ in the above system of equations, we get the following first order equations,

$$F_1'' + F_1' = 0$$
$$G_1'' + G_1' = 0$$
$$H_1'' + P_1 F'_1 H' + P_2 E_1 (F'_1 + G_1^2) = 0$$

with the boundary conditions,

$$F_1 = 0, \quad F'_1 = 1, \quad G_1 = 0, \quad H_1 = 1 \quad \text{at} \quad \xi = 0$$

$$F_1' = 0, \quad G_1 = 0, \quad H_1 = 0 \quad \text{at} \quad \xi \to \infty.$$
Ekman Boundary Layer Mixed Convective Heat Transfer Flow ……

From the first and second order solution, the velocities and temperature fields are obtained as follows,

Primary velocity, \( f'(\eta) = e^{\eta/l} + \varepsilon \left( A_{\eta} e^{-\eta/l} - A_{2\eta} f_{\eta} e^{-\eta/l} - P_{\eta} A_{1} e^{-\eta/l} - 2A_{2\eta} e^{-2\eta/l} \right) \)

Secondary velocity, \( g(\eta) = -2U_0 e E(\eta f_{\eta} e^{-\eta/l}) \)

Temperature, \( \theta(\eta) = (e^{-\eta/l} + A_{1} e^{-2\eta/l}) + \varepsilon \left[ A_{2\eta} e^{-2\eta/l} - \eta f_{\eta} P_{\eta} e^{-\eta/l} + A_{3\eta} e^{-\eta/l} + A_{4\eta} e^{-2\eta/l} + A_{3\eta} (\eta f_{\eta} + A_{3\eta}) e^{-2\eta/l} \right] \)

5. Shear Stress and Nusselt Number

Since the quantities of chief physical interest are shear stress and Nusselt number, hence the primary wall shear stress is defined as,

\[
\tau_{x} = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}
\]

which implies that, \( \tau_{x} \alpha \left[ -1 + \varepsilon \left( -2A_{3\eta} + P_{\eta} A_{1} + 4A_{\eta} \right) \right] \).

The secondary wall shear stress is also defined as,

\[
\tau_{z} = \mu \left( \frac{\partial \omega}{\partial y} \right)_{y=0}
\]

which implies that, \( \tau_{z} \alpha (-2\varepsilon E) \).

As well as the Nusselt number is defined as,

\[
N_{u} = \mu \left( -\frac{\partial T}{\partial y} \right)_{y=0}
\]

which implies that, \( N_{u} \alpha \left[ (-P_{\eta} - 2A_{\eta}) + \varepsilon \left[ -2A_{2\eta} - P_{\eta} - A_{2\eta} (P_{\eta} + 1) - 3A_{3\eta} + A_{3\eta} - 2A_{\eta} A_{2\eta} \right] \right] \).

6. Results and Discussion

In order to discuss the results of the present work, the analytical solutions are obtained by perturbation technique. For investigating the physical situation of the model, we have computed the numerical values of the flow variables (velocities and temperature) for different values of suction parameter, Grashof number, Prandtl number, Eckert number, Ekman number as well as permeability parameter. In this section, the numerical values of the velocities and fluid temperature versus \( \eta \) are plotted in Figures 6.1-6.8.

The primary velocity profiles for different values of \( G_{\eta}, K, P \) and \( E_{\eta} \) are shown in Figures 6.1-6.4. From Figure 6.1, we see that the primary velocity increases with the increase of \( G_{\eta} \). It is observed from Figure 6.2 that the primary velocity decreases with the rise of \( K \) and the same effect of the Prandtl number on the primary velocity is observed from Figure 6.3. An increasing effect on the primary velocity profiles for \( E_{\eta} \) is found from the Figure 6.4.
Figure 6.1. Primary Velocity Profiles for $f_w = 2.0$, $K = 0.1$, $E = 1.0$, $P_r = 7.0$, $E_r = 1.0$.

Figure 6.2. Primary Velocity Profiles for $G_f = 10$, $f_w = 2.0$, $E = 1.0$, $P_r = 7.0$, $E_r = 1.0$.

Figure 6.3. Primary Velocity Profiles for $G_f = 10$, $f_w = 2.0$, $K = 0.1$, $E = 1.0$, $E_r = 1.0$. 
Figure 6.4. Primary Velocity Profiles for $G_i = 10$, $f_w = 2.0$, $K = 0.1$, $E = 1.0$, $P_r = 7.0$.

Figure 6.5. Secondary Velocity Profiles for $G_i = 10$, $K = 0.1$, $E = 1.0$, $P_r = 7$, $E_c = 1.0$.

Figure 6.6. Secondary Velocity Profiles for $G_i = 10$, $f_w = 2$, $K = 0.1$, $P_r = 7$, $E_c = 1.0$.

The secondary velocity curves for different values of $f_w$ and $E$ are displayed in Figures 6.5-6.6. It is observed from the Figure 6.5, the secondary velocity decreases
M.M. Haque, M.S. Uddin, M.A. Islam and M.H. Uddin

with the increase of the Ekman Number. The Figure 6.6 shows that the secondary velocity increases with the rise of $f_w$.

The temperature distributions of fluid are shown in Figures 6.7-6.8. From the two figures, we have observed the temperature decreases with the increases of $E_c$ as well as the temperature rises in case of strong Prandtl number.

![Figure 6.7. Temperature Profiles for $G_r = 10$, $f_w = 2.0$, $K = 0.1$, $E = 1.0$, $E_c = 1.0$.](image)

![Figure 6.8. Temperature Profiles for $G_r = 10$, $f_w = 2.0$, $K = 0.1$, $E = 1.0$, $P_r = 7.0$.](image)

To discuss the quantities of chief physical interest of the problem, the numerical values of primary shear stress ($\tau_x$), secondary shear stress ($\tau_z$) and Nusselt number ($N_u$) are tabulated in the following Table 6.1 due to the variation in $G_r$, $f_w$, $K$, $E$, $P_r$ and $E_c$ for an externally cooled ($G_r > 0$) plate.

It is observed from Table 6.1, the primary shear stress at the wall increases in case of strong of $G_r$ or $E$, while it decreases with the rise of $P_r$, $f_w$ or $K$ but it remains unchanged with the change of $E$. We see that, the secondary shear stress at the wall
Ekman Boundary Layer Mixed Convective Heat Transfer Flow …

increases for rising the \( w_f \) while it decreases with the increase of \( E \) but no effect is found by the change of \( G_r, P_r, K \) or \( E_c \). Also the Nusselt number increases in case of strong \( G_r, f_v, K \) or \( E_c \) while it decreases the rise of Prandtl number but it remains unchanged with the change of Ekman number.

**Table 6.1. Numerical Values of Shear Stresses \( (\tau_x, \tau_z) \) with Nusselt Number**

<table>
<thead>
<tr>
<th>S. No.</th>
<th>( G_r )</th>
<th>( f_v )</th>
<th>( K )</th>
<th>( E )</th>
<th>( P_r )</th>
<th>( E_c )</th>
<th>( \tau_x )</th>
<th>( \tau_z )</th>
<th>( N_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>10.0</td>
<td>2.0</td>
<td>0.1</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.217</td>
<td>-0.500</td>
<td>3.871</td>
</tr>
<tr>
<td>2.</td>
<td>11.0</td>
<td>2.0</td>
<td>0.1</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.271</td>
<td>-0.500</td>
<td>4.367</td>
</tr>
<tr>
<td>3.</td>
<td>12.0</td>
<td>2.0</td>
<td>0.1</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.325</td>
<td>-0.500</td>
<td>4.863</td>
</tr>
<tr>
<td>4.</td>
<td>10.0</td>
<td>2.1</td>
<td>0.1</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.157</td>
<td>-0.454</td>
<td>4.292</td>
</tr>
<tr>
<td>5.</td>
<td>10.0</td>
<td>2.2</td>
<td>0.1</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.112</td>
<td>-0.413</td>
<td>4.657</td>
</tr>
<tr>
<td>6.</td>
<td>10.0</td>
<td>2.0</td>
<td>0.2</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.204</td>
<td>-0.500</td>
<td>4.011</td>
</tr>
<tr>
<td>7.</td>
<td>10.0</td>
<td>2.0</td>
<td>0.3</td>
<td>1.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.192</td>
<td>-0.500</td>
<td>4.151</td>
</tr>
<tr>
<td>8.</td>
<td>10.0</td>
<td>2.0</td>
<td>0.1</td>
<td>2.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.217</td>
<td>-1.000</td>
<td>3.871</td>
</tr>
<tr>
<td>9.</td>
<td>10.0</td>
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<td>0.1</td>
<td>3.0</td>
<td>7.0</td>
<td>1.0</td>
<td>0.217</td>
<td>-1.500</td>
<td>3.871</td>
</tr>
<tr>
<td>10.</td>
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<td>5.0</td>
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<td>-0.500</td>
<td>6.931</td>
<td></td>
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<tr>
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<td>1.0</td>
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<tr>
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<td>1.0</td>
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</tr>
<tr>
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<td>1.0</td>
<td>7.0</td>
<td>1.2</td>
<td>0.304</td>
<td>-0.500</td>
<td>5.135</td>
</tr>
</tbody>
</table>

7. Conclusions
Some of the important findings obtained from the graphical representation of the results with the numerical values of table are listed below;

1. The primary fluid velocity increases with the increase of \( G_r \) or \( E_c \) while it decreases with the increase of \( P_r \) or \( K \).
2. The secondary velocity of fluid increases with the increase of \( f_v \) while it decreases with the increase of \( E \).
3. The fluid temperature is increasingly affected by \( P_r \) and decreasingly affected by \( E_c \).
4. The primary shear stress at the wall increases in case of strong \( G_r \) or \( E_c \) while decreases with the increase of \( P_r, f_v \) or \( K \).
5. The secondary shear stress at the wall increases in case of strong \( f_v \) while decreases with the increase of \( E \).
6. The Nusselt number increases with the increase of \( G_r, f_v, K \) or \( E_c \) while it decreases in case of strong Prandtl number.
Appendix

\[ A_{11} = \frac{E_c}{2P_{\infty}}, \quad A_{12} = 1 + \kappa, \quad A_{21} = \frac{G_r}{P_r^2(P_r - 1)}, \quad A_{22} = \left( \frac{1 + G_r A_{11}}{4} \right), \]

\[ A_{33} = P_r \left( 2A_{11} - 4A_{21}E_r \right), \quad A_{41} = P_r^2 \left( 2P_r E_r A_{11} - 1 \right), \quad A_{42} = P_r \left( 8A_{21}E_r - 2A_{11} \right), \quad A_{43} = 2A_{32}P_r E_r, \]

\[ A_{44} = \frac{A_{33} - A_{41}}{4 - 2P_r}, \quad A_{45} = \frac{A_{42}}{1 + P_r}, \quad A_{46} = \frac{A_{43}}{9 - 3P_r}, \quad A_{47} = \frac{A_{44}}{4 - 2P_r}, \quad A_{48} = \frac{A_{45}}{4 - 2P_r}, \quad A_{49} = \frac{A_{46}}{4 - 2P_r}. \]

REFERENCES


