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Further Results on Divisor Cordial Labeling

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Abstract. A divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, ... |V|\}$ such that an edge uv is assigned the label 1 if $f(u) | f(v)$ or $f(v) | f(u)$ and the label 0 otherwise, then number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a divisor cordial labeling is called a divisor cordial graph. In this paper we prove that helm H_n, flower graph Fl_n and Gear graph G_n are divisor cordial graphs. Moreover we show that switching of a vertex in cycle C_n, switching of a rim vertex in wheel W_n and switching of the apex vertex in helm H_n admit divisor cordial labeling.

Keywords: Labeling, Divisor cordial labeling, Switching of a vertex

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen [1] while for number theory we refer to Burton [2] . We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 1.1. If the vertices are assigned values subject to certain condition(s) then it is known as *graph labeling*.

According to Beineke and Hegde [3] graph labeling serves as a frontier between number theory and structure of graphs. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. Yegnanaryanan and Vaidhyanathan [4] have discussed applications of edge balanced graph labeling, edge magic labeling and (1,1) edge magic graphs. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [5].

Definition 1.2. A mapping $f:V(G)\rightarrow\{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.

Notation 1.3. If for an edge $e=uv$, the induced edge labeling $f^*:E(G)\rightarrow\{0,1\}$ is given by $f^*(e)=|f(u)-f(v)|$. Then

$v_f(i)=$ number of vertices of G having label i under f

$e_f(i)=$ number of vertices of G having label i under f^*

Definition 1.4. A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_f(0)-v_f(1)|\leq 1$ and $|e_f(0)-e_f(1)|\leq 1$. A graph G is *cordial* if it admits cordial labeling.

The above concept was introduced by Cahit [6]. After this many labeling schemes are also introduced with minor variations in cordial theme. The product cordial labeling, total product cordial labeling and prime cordial labeling are among mention a few. The present work is focused on divisor cordial labeling.

Definition 1.5. A *prime cordial labeling* of a graph G with vertex set $V(G)$ is a bijection $f:V(G)\rightarrow\{1,2,3,\ldots,|V(G)|\}$ and the induced function $f^*:E(G)\rightarrow$ $\{0,1\}$ is defined by

$$f^*(e=uv)=\begin{cases}1 & \text{, if } \gcd(f(u),f(v))=1;\\ 0 & \text{, otherwise.}\end{cases}$$

which satisfies the condition $|e_f(0)-e_f(1)|\leq 1$. A graph which admits prime cordial labeling is called a *prime cordial graph*.

The concept of prime cordial labeling was introduced by Sundaram *et al.* [7] and in the same paper they have investigated several results on prime cordial labeling. Vaidya and Vihol [8, 9] as well as Vaidya and Shah [10, 11, 12] have proved many results on prime cordial labeling.

Motivated through the concept of prime cordial labeling Varatharajan *et al*. [13] introduced a new concept called divisor cordial labeling which is a combination of divisibility of numbers and cordial labelings of graphs.

Definition 1.6. Let $G=(V(G),E(G))$ be a simple graph and $f:\rightarrow\{1,2,\ldots|V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u)\mid f(v)$ or $f(v)\mid f(u)$ and the label 0 otherwise. The function f is called a *divisor cordial labeling* if $|e_f(0)-e_f(1)|\leq 1$. A graph with a divisor cordial labeling is called a *divisor cordial graph.*

In [13] authors have proved that path, cycle, wheel, star, $K_{2,n}$ and $K_{3,n}$ are divisor cordial graphs while K_n is not divisor cordial for $n \geq 7$. The divisor cordial labeling of full binary tree as well as some star related graphs are reported by Varatharajan *et al*. [14] while some star and bistar related graphs are proved to be divisor cordial graphs by Vaidya and Shah [15].

It is important to note that prime cordial labeling and divisor cordial labeling are two independent concepts. A graph may possess one or both of these properties or neither as exhibited below.

i) P_n ($n \geq 6$) is both prime cordial as proved in [7] and divisor cordial as proved in [13].
ii) C_3 is not prime cordial as proved in [7] but it is divisor cordial as proved in [13].
iii) We found that a 7-regular graph with 12 vertices admits prime cordial labeling but does not admit divisor cordial labeling.
iv) Complete graph K_7 is not a prime cordial as stated in Gallian [5] and not divisor cordial as proved in [13].

Definition 1.7. The *helm* H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices.

Definition 1.8. The *flower* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm. It contains three types of vertices: an apex of degree $2n$, n vertices of degree 4 and n vertices of degree 2.

Definition 1.9. Let $e=uv$ be an edge of the graph G and w is not a vertex of G. The edge e is called *subdivided* when it is replaced by edges $e' = uw$ and $e'' = wv$.

Definition 1.10. The *gear graph* G_n is obtained from the wheel by subdividing each of its rim edge.

Definition 1.11. A *vertex switching* G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

2. Divisor Cordial Labeling of Some Wheel Related Graphs

Theorem 2.1. H_n is a divisor cordial graph for every n.

Proof : Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the pendant vertices of H_n. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define vertex labeling as $f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n+1\}$ as follows.

$f(v) = 1,$

For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil = k$,

Assign the labels v_i and u_i such that $2f(v_i) = f(u_i)$ and $f(v_i) \nmid f(v_{i+1})$.
Now for remaining vertices, $v_{k+1}, v_{k+2}, \ldots, v_n$ and $u_{k+1}, u_{k+2}, \ldots, u_n$ assign the labels such that $f(v_j) \nmid f(v_{j+1})$ where $k \le j \le n-1$, $f(v_n) \nmid f(v_1)$ and $f(v_j) \nmid f(u_j)$ where $k < j \le n$.
In view of above labeling pattern we have, $e_f(1) = \left\lceil \frac{3n}{2} \right\rceil, e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor$.
Thus, $|e_f(0) - e_f(1)| \le 1$.
Hence, H_n is a divisor cordial graph for each n.

Example 2.2. The graph H_{13} and its divisor cordial labeling is shown in Figure 1.

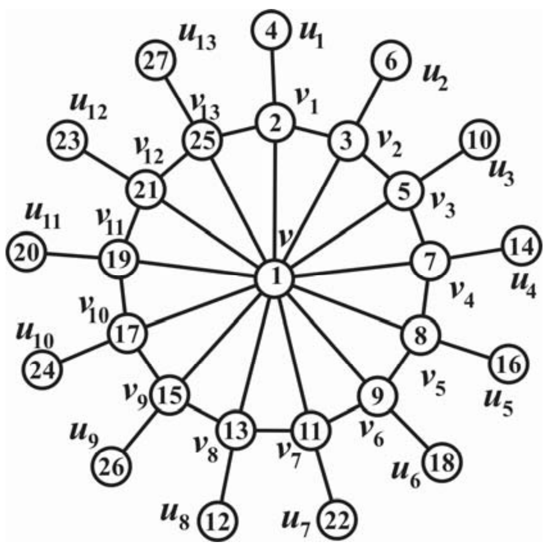

Figure 1: Divisor cordial labeling of H_{13}

Theorem 2.3. Fl_n is a divisor cordial graph for each n.
Proof : Let v be the apex $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 of Fl_n. Then $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$. We define vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, 2n+1\}$ as follows.

$f(v) = 1,\qquad f(v_1) = 2,$

$f(u_1) = 3,$

$f(v_{1+i}) = 5 + 2(i-1);\quad 1 \le i \le n-1$

$f(u_{1+i}) = 4 + 2(i-1);\quad 1 \le i \le n-1$

In view of the above labeling pattern we have, $e_f(0) = 2n = e_f(1)$.
Thus, $|e_f(0) - e_f(1)| \le 1$.
Hence, Fl_n is a divisor cordial graph for each n.

Example 2.4. Divisor cordial labeling of the graph Fl_{11} is shown in Figure 2.

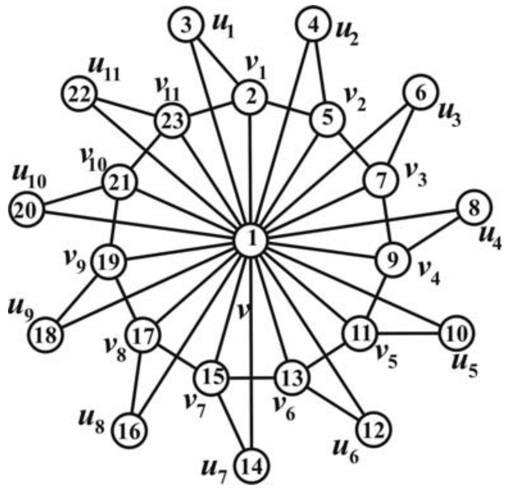

Figure 2: Divisor cordial labeling of Fl_{11}

Theorem 2.5. G_n is a divisor cordial graph for every n.

Proof: Let W_n be the wheel with apex vertex v and rim vertices $v_1, v_2, \ldots, v_n$. To obtain the gear graph G_n subdivide each rim edge of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each u_i is added between v_i and v_{i+1} for $i = 1, 2, \ldots, n-1$ and u_n is added between v_1 and v_n. Then $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. We define vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, 2n+1\}$, as follows.

Our aim is to generate $\left\lceil \frac{3n}{2} \right\rceil$ edges having label 1 and $\left\lfloor \frac{3n}{2} \right\rfloor$ edges having label 0.

$f(v) = 1$, which generates n edges having label 1.

Now it remains to generate $k = \left\lceil \frac{3n}{2} \right\rceil - n$ edges with label 1.

For the vertices $v_1, u_1, v_2, u_2, \ldots$ assign the vertex label as per following ordered pattern upto it generate k edges with label 1.

$$\begin{array}{ccccc}
2, & 2^2, & 2^3, & \ldots, & 2^{k_1}, \\
3, & 3\times 2, & 3\times 2^2, & \ldots, & 3\times 2^{k_2}, \\
5, & 5\times 2, & 5\times 2^2, & \ldots, & 5\times 2^{k_3}, \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots, \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots,
\end{array}$$

where $(2m-1)2^{k_m} \le 2n+1$ and $m \ge 1, k_m \ge 0$.

Observe that $(2m-1)2^{\alpha} \mid (2m-1)2^{\alpha+1}$ and $(2m-1)2^{k_i}$ does not divide $2m+1$. Then for remaining vertices of G_n, assign the vertex label such that the consecutive vertices do not generate edge label 1.

In view of above labeling pattern we have, $e_f(1) = \left\lceil \frac{3n}{2} \right\rceil, e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, G_n is a divisor cordial graph for each n.

Example 2.6. Divisor cordial labeling of the graph G_{20} is shown in Figure 3.

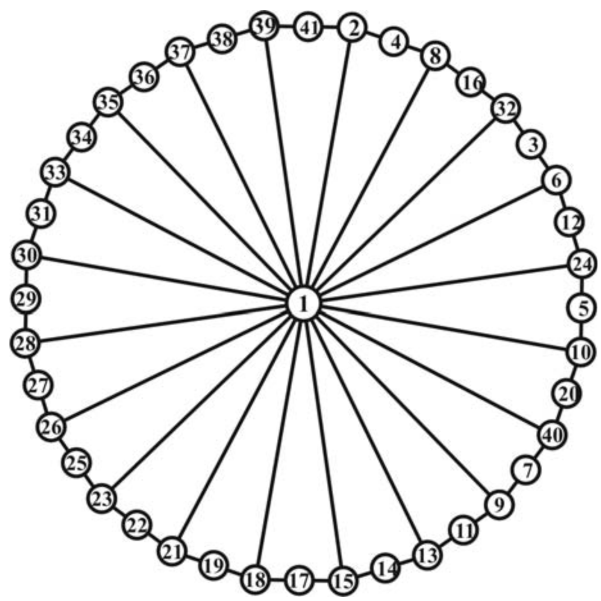

Figure 3: Divisor cordial labeling of G_{20}

3. Switching of a Vertex and Divisor Cordial Labeling

Theorem 3.1. Switching of a vertex in cycle C_n admits divisor cordial labeling.

Proof: Let $v_1, v_2, \ldots, v_n$ be the successive vertices of C_n and G_v denotes graph obtained by switching of vertex v of $G = C_n$. Without loss of generality let the switched vertex be v_1.

We note that $\left|V\left(G_{v_1}\right)\right| = n$ and $\left|E\left(G_{v_1}\right)\right| = 2n-5$. We define vertex labeling

$f : V\left(G_{v_1}\right) \to \{1, 2, \ldots, n\}$ as follows:

$f(v_1) = 1,\ f(v_{1+i}) = 1+i$.

In view of the above labeling pattern we have, $e_f(1) = n-3, e_f(0) = n-2$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by switching of a vertex in cycle C_n is a divisor cordial labeling.

Example 3.2. The graph obtained by switching of a vertex in cycle C_8 and its divisor cordial labeling is shown in Figure 4.

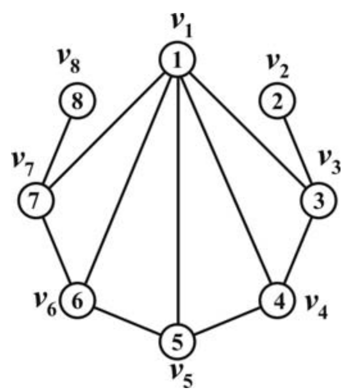

Figure 4: Switching of a vertex in C_8 and its divisor cordial labeling

Theorem 3.3. Switching of a rim vertex in a wheel W_n admits divisor cordial labeling.

Proof : Let v be the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel W_n. Let G_{v_1} denotes graph obtained by switching of a rim vertex v_1 of $G = W_n$. We note that $\left|V\left(G_{v_1}\right)\right| = n+1$ and $\left|E\left(G_{v_1}\right)\right| = 3n-6$. To define vertex labeling $f : V\left(G_{v_1}\right) \to \{1, 2, \ldots, n+1\}$, we consider following two cases.

Case 1: $n = 4$

$f(v) = 1, f(v_1) = 5, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4$. Then $e_f(0) = 3 = e_f(1)$.

Case 2: $n \geq 5$

$f(v) = 2$, $f(v_1) = 1$,

$f(v_2) = 3$, $f(v_3) = 6$,

$f(v_4) = 4$, $f(v_5) = 5$,

$f(v_{5+i}) = 6 + i;\quad 1 \leq i \leq n-5$

In view of the above defined labeling pattern for case 2,

If n is even then $e_f(0) = \frac{3n-6}{2} = e_f(1)$, otherwise $e_f(0) = \left\lfloor \frac{3n-6}{2} \right\rfloor = e_f(1) - 1$.

Thus in both the cases we have, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by switching of a rim vertex in a wheel W_n is a divisor cordial labeling.

Example 3.4. The graph obtained by switching of a rim vertex in the wheel W_9 and its divisor cordial labeling is shown in Figure 5.

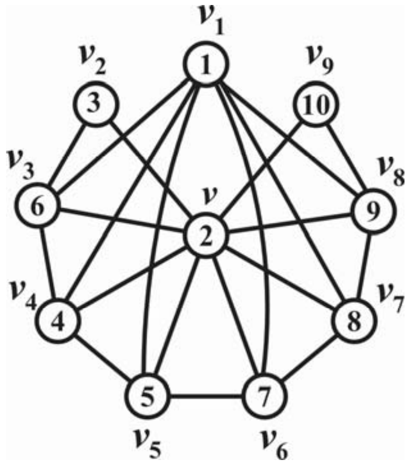

Figure 5: Switching of a rim vertex in W_9 and its divisor cordial labeling

Theorem 3.5. Switching of the apex vertex in helm H_n admits divisor cordial labeling.

Proof: Let v be the apex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 and $u_1, u_2, \ldots, u_n$ be the pendant vertices of H_n. Let G_v denotes graph obtained by switching of an apex vertex v of $G = H_n$. We note that $\left|V\left(G_{v_1}\right)\right| = 2n+1$ and $\left|E\left(G_{v_1}\right)\right| = 3n$. We define vertex labeling $f : V\left(G_{v_1}\right) \to \{1, 2, \ldots, 2n+1\}$ as follows:

Our aim is to generate $\left\lceil \frac{3n}{2} \right\rceil$ edges having label 1 and $\left\lfloor \frac{3n}{2} \right\rfloor$ edges having label 0.

$f(v) = 1$, which generates n edges having label 1.

Now it remains to generate $k = \left\lceil \frac{3n}{2} \right\rceil - n$ edges with label 1.

For the vertices $v_1, v_2, \ldots, v_l$ assign the vertex label as per following ordered pattern upto it generate k edges with label 1.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & \ldots, & 2^{k_1}, \\ 3, & 3\times 2, & 3\times 2^2, & \ldots, & 3\times 2^{k_2}, \\ 5, & 5\times 2, & 5\times 2^2, & \ldots, & 5\times 2^{k_3}, \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, \end{array}$$

where $(2m-1)2^{k_m} \leq 2n+1$ and $m \geq 1, k_m \geq 0$.

Observe that $(2m-1)2^{\alpha} \mid (2m-1)2^{\alpha+1}$ and $(2m-1)2^{k_i}$ does not divide $2m+1$.

Then for remaining vertices $v_{l+1}, v_{l+2}, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ assign the vertex label such that no edge label generate 1.

In view of above labeling pattern we have, $e_f(1) = \left\lceil \frac{3n}{2} \right\rceil, e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by switching of the apex vertex in helm H_n admits divisor cordial labeling.

Example 3.6. The graph obtained by switching of the apex vertex in helm H_{11} and its divisor cordial labeling is shown in Figure 6.

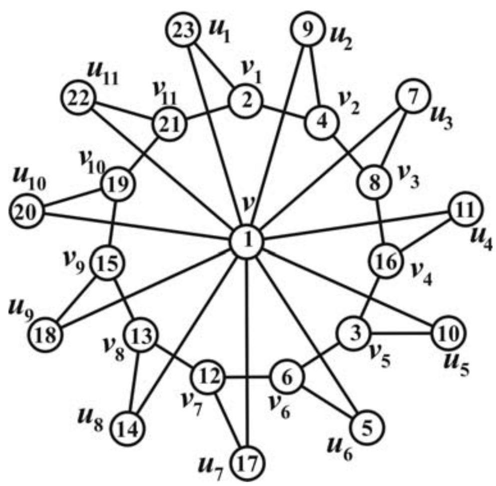

Figure 6: Switching of the apex vertex in H_{11} and its divisor cordial labeling

4. Concluding Remarks

The divisor cordial labeling is a variant of cordial labeling. It is very interesting to investigate graph or graph families which are divisor cordial as all the graphs do not admit divisor cordial labeling. Here it has been proved that helm H_n, flower graph Fl_n and Gear graph G_n are divisor cordial graphs. The graphs C_n and W_n are proved to be divisor cordial graphs by Varatharajan *et al.* [13] while we prove the graphs obtained by switching of a vertex in C_n, switching of a rim vertex in W_n and switching of the apex vertex in H_n are divisor cordial graphs. Hence C_n, W_n and H_n are switching invariant graphs for divisor cordial labeling.

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