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Some New Families of Divisor Cordial Graph

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Abstract. A divisor cordial labeling of a graph G with vertex set V vertex G is a bijection f from V to {1, 2, 3, . . . |V|} such that an edge uv is assigned the label 1 if f(u) divides f(v) or f(v)divides f(u) and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a divisor cordial labeling, then it is called divisor cordial graph. In this paper we prove that flower graphs and helm graphs are divisor cordial. We also prove some special graphs such as switching of a vertex of cycle, wheel, helm; duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle; split graph of $K_{1,n}$, $B_{n,n}$, $B_{n,n}^2$ are divisor cordial graphs.

Keywords: Cordial labeling, divisor cordial labeling, divisor cordial graph, split graph, Bistar

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

By a graph, we mean a finite undirected graph without loops and multiple edges. For terms not defined here we refer to Harary [2].

Definition 1.1. [8] A binary vertex labeling of graph is called a ***cordial labeling*** if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$, where $v_f(i)$ denote the number of vertices labeled with i under f and $e_f(i)$ denote the number of edges labeled with i, where i = 0, 1. A graph G is called cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit. Many researchers have studied cordiality of graphs. Cahit [8] proved that tree is cordial. In the same paper, he proved that K_n is cordial if and only if $n \le 3$. Ho et al. [3] proved that unicycle graph is cordial unless it is $C_{4k+2.}$ Vaidya et. Al. [6] has also discussed the cordiality of various graphs.

Definition 1.2. [8] Let G be a graph with vertex set v(G) and edge set E(G) and let f: $E(G) \rightarrow \{0,1\}$. Define f* on V(G) by $f^*(v) = \sum \{ f(uv), uv \in E(G) \} \ (\text{mod}2)$. The function f is called an ***E-cordial labeling*** of G if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph is called ***E-cordial*** if it admits E-cordial labeling.

In 1997 Yilmaz and Cahit [2] have introduced E-cordial labeling as a weaker version of edge – graceful labeling. They proved that the trees with n vertices, K_n, C_n are E-cordial if and only if $n \perp 2$ (mod4) while $K_{m,n}$ admits E-cordial labeling if and only if $m+n \perp 2$ (mod4).

Definition 1.3. [4] A prime cordial labeling of a graph G with vertex set V is a bijection f from V to {1, 2, 3, . . . , |V|} such that if each edge uv is assigned the label 1 if gcd (f(u), f(v)) = 1 and 0 if gcd (f(u), f(v)) >1, then the number of edges with 1 differ by at most 1.

Sundaram et.al [4] has introduced the notion of prime cordial labeling. They proved the following graph as prime cordial labeling – C_n if and only if $n \geq 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$ (n, odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bi stars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders. J. Babujee and L. Shobana [3] proved the existence of prime cordial labeling for sun graph, kite graph and coconut tree and Y -tree, < $K_{1,n}$: 2 > $n \geq 1$). Hoffman tree, and $K_2 \Theta C_n (C_n)$.

Definition 1.4. [10] Let G = (V, E) be a simple graph and f : V → {1, 2, 3, . . . |V|} be a bijection. For each edge uv, assign the label 1 if either f(u) / f(v) or f(v) /f(u) and the label 0 if f(u)∤f(v). Then f is called a ***divisor cordial labeling***. A graph with a divisor cordial labeling is called ***divisor cordial graph.***

Varatharajan et al. [11], introduced the concept of divisor cordial and proved the graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial graphs and in [12], they proved some special classes of graphs such as full binary tree, dragon, corona, $G * K_{2,n}$ and $G * K_{3,n}$ are divisor cordial.

Labeled graph have variety of application in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal auto correlation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication networks and to determine optimal circuit layouts.

In this paper, we prove that flower graph, Helm graph are divisor cordial. We also prove some special graphs such as switching of a vertex of cycle, wheel, helm; duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle; split graph of $K_{1,n}$, $B_{n,n}$, $B_{n,n}^2$ are also divisor cordial graphs. Before we deal with the main results, we give some definitions that are useful to the ensuing sections.

Definition 1.5. [8] The ***Helm, H_n*** is the graph on 2n + 1 vertices, obtained from a wheel W_n by attaching a pendant edge to each of n rim vertices.

Definition 1.6. [8] The ***closed helm CH_n*** is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.

Definition 1.7. [9] The ***flower graph Fl_n*** is the graph obtained from a helm H_n by joining each pendant vertex to the apex vertex of the helm.

2. Main Results

Theorem 2.1. Fl_n is divisor cordial for $n \geq 3$.

Proof: Let H_n be a helm with v as the apex vertex; $v_1, v_2, \ldots v_n$ as the vertices of the cycle and $u_1, u_2, u_3, \ldots u_n$ be the pendant vertices. Let Fl_n be the flower graph obtained from helm H_n then $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)|=4n$. We define f: $V(Fl_n) \rightarrow \{1, 2, 3, \ldots |V|\}$ as,

$f(v) = 1$

$f(v_1) = 2$

$f(u_1) = 3$

$f(v_i) = 2i+1$; $2 \leq I \leq n$

$f(u_i) = 2i$; $2 \leq i \leq n$

From the above, we observe that since the apex vertex v is labeled with 1, so the edges incident with v receive label 1 and there are 2n edges incident to v. Therefore $e_f(1) = 2n$. The edges u_iv_i receive the label 0 for every I = 1, 2, 3, . . . , n. Since $f(u_i)$ and $f(v_i)$ are consecutive labels. Similarly, all the edges in the cycle of the base wheel also receive the label 0 since the vertices of the cycle are labeled with consecuti9ve odd integers except for v_1, where $f(v_1) = 2$. Thus $e_f(0) = 2n$.

Therefore $|e_f(0) - e_f(1)| = 0$

Hence Fl_n is divisor cordial graph

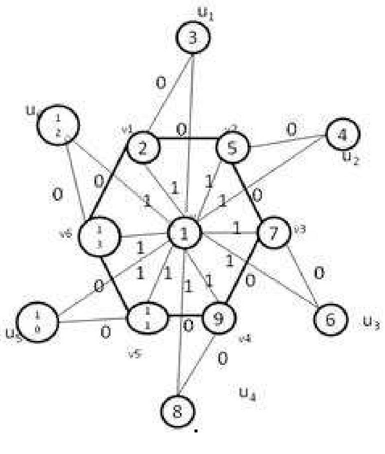

Figure 1: Fl_6 is a divisor cordial

Theorem 2.2. Helm graph H_n is divisor cordial, for $n > 3$.

Proof: Let v be the apex vertex; $v_1, v_2, \ldots v_n$ are the vertices of cycle and $u_1, u_2, \ldots u_n$ be the pendant vertices for n>3. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define f as follows,

$f(v) = 1$

$f(u_i) = 2i + 1$; $1 \leq i \leq n-1$

$f(u_n) = f(v_{n-1}) + 4$

$f(v_n) = f(u_{n-1}) + 2$

The vertices v_i are labeled in the following order,

$2, 2 \times 2, 2 \times 2^2, \ldots 2 \times 2^{k1}$

$6, 6 \times 2, 6 \times 2^2, \ldots 6 \times 2^{k2}$ (1)

$10, 10 \times 2, 10 \times 2^2, \ldots 10 \times 2^{k3}$

. . . .

where $(4m - 2)\, 2^{k_m} \leq n$ and $m \geq 1$, $k_m \geq 0$. We observe that $(4m - 2)\, 2^a$ divides $(4m - 2)\, 2^b$; ($a< b$) and $(4m - 2)\, 2^{k_i}$ does not divide $(4m + 2)$.

Case 1. n is even

The apex vertex v is labeled as 1, so the n edges incident to v receive label 1. Now the cyclic vertices v_i being labeled as in (1), and so (n/2) edges receive the label 1 and (n/2) edges receive 0. Since $f(u_i) = 2i+1$; $1 \leq i \leq n-1$ and v_i's are labeled as in (1), they does not divide each other, and receives label 0.

Therefore $e_f(1) = e_f(0) = n + n/2 = 3n / 2$

Thus $|e_f(0) - e_f(1)| = 0$

Case 2. n is odd

The edges vv_i receive the label 1, as given in the above case, and so n edges get the label 1. The cyclic vertices v_i are labeled as in (1), so that (n-1)/2 edges receive label 1 and (n + 1) / 2 edges receive label 0. The rim edges v_iv_j receive label 0 (as in the above case).

Therefore $e_f(1) = n + (n + 1) / 2 = (3n + 1) / 2$

$e_f(0) = n + (n – 1) / 2 = (3n – 1) / 2$

Thus from the two cases $| e_f(0) - e_f(1) | \leq 0$

Hence H_n is divisor cordial graph

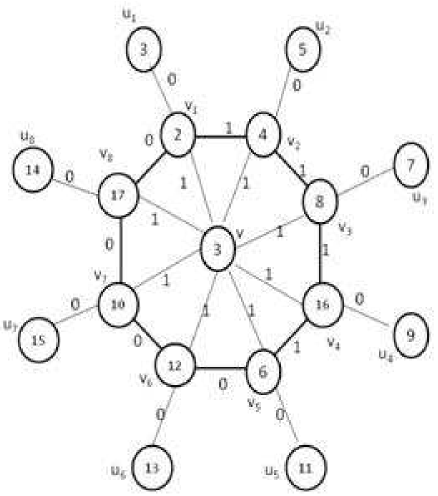

Figure 2. H_8 is divisor cordial

3. Divisor Cordial graphs obtained by switching of a vertex.

Definition 3.1. [8] A *vertex switching G_v* of a graph G is the graph obtained by taking a vertex v of G, removing the entire edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

Theorem 3.2. The graph obtained by switching of an arbitrary vertex in cycle C_n is divisor cordial.

Proof: Let $v_1, v_2,\ldots,v_n$ be the successive vertices of C_n, and G_v denotes the graph obtained by switching of vertex *v* of G. Without loss of generality let the switched vertex be v_1. We note that $|V(G_{v1})|= n$ and $|E(G_{v1})|= 2n - 5$. We define $f : V(G_{v1}) \rightarrow \{1, 2,3, \ldots, n\}$ as follows:

$f(v_1) = 1$

$f(v_i) = i \; ; 2 \leq i \leq n$

Since the switched vertex v_1 is labeled as 1, the (n - 5) edges incident to 1 receive label 1 and other edges receives 0 as the consecutive integers does not divides each other. Therefore $e_f(0) = n - 2$ and $e_f(1) = n - 3$

Thus $| e_f(0) - e_f(1) | = 1$

Hence the graph obtained by switching of an arbitrary vertex in cycle C_n is a divisor cordial graph.

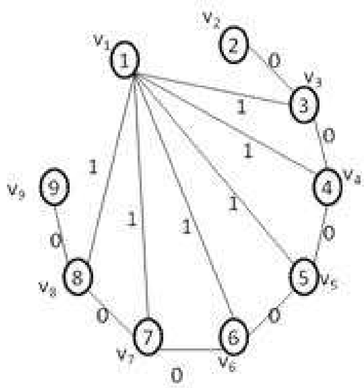

Figure 3. Switching of v_1 in cycle C_7

Theorem 3.3. The graph obtained by switching of a rim vertex in wheel W_n, $n \geq 4$ is a divisor cordial graph.

Proof: Let v be the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel W_n .Let G_{v1} denote graph obtained by switching of a rim vertex v_1 of $G = W_n$. We note that $|V (G_{v1})| = n + 1$ and $|E (G_{v1})| = 3(n - 2)$. We define f as follows,

$f(v) = 1; \quad f(v_1) = 2 \; ; \quad f(v_2) = 4;$

$f(v_3) = 3; \quad f(v_i) = i+1 \; ; 4 \leq i \leq n.$

Case 1. n is odd

The apex vertex v is labeled with 1. Trivially the $(n - 1)$ edges adjacent to it receive label 1. Also $f(v_1) = 2$ and $f(v_i) = i+1$; $4 \leq i \leq n$. Out of (n-3) edges incident to v_1, (n-5) / 2 edges receive label 1 and other (n-1) / 2 receive 0. Also the consecutive numbers does not divide each other, so the (n - 2) edges v_iv_j receive label 0.

Therefore, $e_f(1) = (n-1)+ (n-5) /2 = (3n -7)/2$ and

$e_f(0) = (n-2) + (n-1) / 2 = (3n-5)/2$.

Case 2. n is even

The apex vertex is 1, trivially the (n – 1) edges receive 1. Since $f(v_1) = 2$, out of (n – 2) edges, (n - 4)/2 edges receive 1 and (n -2)/2 edges gets label 0. Also, since the consecutive numbers does not divide each other the (n – 2) edges v_iv_j contribute 0.
Therefore $e_f(1) = (n-1) + (n-4)/2 = 3(n-2)/2$ and

$e_f(0) = (n-2) + (n-4)/2 = 3(n-2)/2$.

Thus $| e_f(0) - e_f(1) | \leq 0$.
Hence G is a divisor cordial graph.

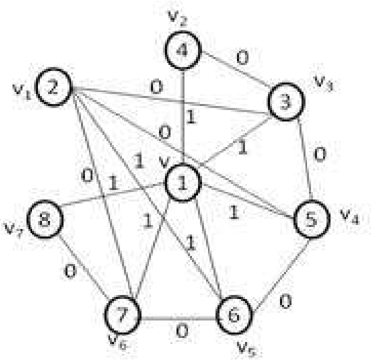

Figure 4. Switching of v_1 in wheel W_7

Theorem 3.4. The graph obtained by switching of an apex vertex in H_n admits divisor cordial labeling.
Proof: Let H_n be a helm with v as the apex vertex; $v_1, v_2, v_3, \ldots v_n$ be the vertices of cycle and $u_1, u_2, u_3, \ldots u_n$ be the pendant vertices. Let G_v denotes graph obtained by switching of an apex vertex V of $G = H_n$. Here $|V(H_n)| = 2n + 1$.
We define f as follows.

$f(v) = 1$

$f(u_j) = 2j + 1; 1 \leq j \leq n$

$f(v_n) = 2n + 1$

$f(u_n) = 2n$

Label the vertices v_i are arranged as in (1) where $(4m-2)\, 2^{km} \leq n$ and $m \geq 1$, $k_m \geq 0$. We observe that $(4m - 2)\, 2^a$ divides $(4m - 2)2^b$; $(a < b)$ and $(4m - 2)\, 2^{ki}$ does not divide (4m + 2).

Case 1. n is odd
The apex vertex v is labeled with 1, trivially edges incident to v receive label 1, and there are n such 1's. Since the vertices v_i are arranged as in (1), the adjacent vertices which are their multiplier's divide each other, and so the edges incident to it receive 1, and there are (n - 1)/2 such 1's and so other (n + 1)/2 edges receive 0. Now the edges u_iv_i contribute 0. Since u_i and v_i are consecutive numbers they does not divide each other. Therefore, $e_f(1) = n + (n - 1)/2 = (3n - 1)/2$ and $e_f(0) = n + (n + 1)/2 = (3n + 1)/2$.

Case 2. n is even
The vertices v_i are labeled as in (2), so that the outer cycle n/2 edges will receive 1, all the other arguments are as in the above case. Therefore $e_f(0) = e_f(1) = n + n/2 = 3n/2$.
Hence in both the cases, $| e_f(0) - e_f(1) | \leq 1$.

Hence the graph obtained by switching of H_n is a divisor cordial graph.

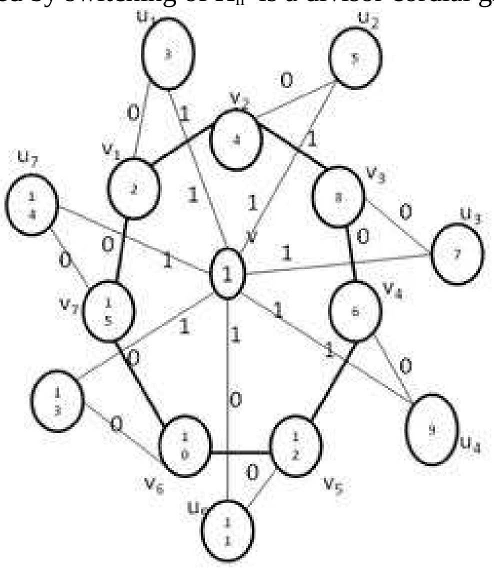

Figure 5. Switching of apex vertex *v* in graph H_7

4. Duplication of a vertex and duplication of an edge

Definition 4.1. [5] ***Duplication of a vertex v_k*** graph G produces a new graph G, by adding a new vertex v_k' in such a way that $N(v_k) = N(v_k')$

Definition 4.2. [5] ***Duplication of an edge* e_k** by an edge e_k' produces a new graph G, in such a way that $N(v_k) \cap N(v_k') = \{v_{k-1}\}$ and $N(v_{k+1}) \cap N(v_{k+1}') = \{v_{k+2}\}$.

Definition 4.3. [5] For a graph G, the split graph is obtained by duplicating its vertices altogether.

Theorem 4.4. The graph obtained by duplication of an arbitrary vertex of C_n admits divisor cordial labeling.

Proof: Let $v_1, v_2, v_3 \ldots v_n$ be the vertices of the cycle C_n.

Let G be the graph obtained by duplicating an arbitrary vertex of C_n.. Without loss of generality, let this vertex be v_1 . Then $E(G) = \{E(C_n.); e', e''\}$ where $e' = v_1'v_2$ and $e'' = v_nv_1'$ and $V(G) = \{V(C_n.), v_1'\}$. Hence $|V(G)| = n + 1$, $|E(G)| = n + 2$.

The other vertices are labeled in the following order.

$$\begin{array}{l} 1, 2, 2 \times 2, 2 \times 2^2, \ldots 2 \times 2^{k1} \\ 3, 3 \times 2, 3 \times 2^2, \ldots 3 \times 2^{k2} \\ 5, 5 \times 2, 5 \times 2^2, \ldots 5 \times 2^{k3} \\ \ldots \ldots \ldots \end{array} \tag{2}$$

where $(2m - 1)\, 2^{km} \le n$ and $m \ge 1$, $k_m > 0$. We observe that $(2m - 1)\, 2^a$ divides $(2m - 1)\, 2^b$; $(a < b)$ and $(2m - 1)\, 2^{ki}$ does not divide $(2m + 1)$.

Case 1. n is even

We label the vertices as follows.

$f(v_1) = 1$; $f(v_1') = n + 1$

The cycle is labeled as in (2), so that $(n + 2)/2$ edges receive the label 1 and $(n - 2)/2$ edges receive the label 0. The two edges $v_1'v_2$ and $v_1'v_n$ contribute 0.

Therefore $e_f(1) = (n + 2)/2$ and $e_f(0) = (n - 2)/2 + 2 = (n + 2)/2$.

Case 2. n is odd

We label the vertices as follows.

$f(v_1) = 1 : f(v_1') = n$

The vertices in the cycle are labeled as in (2). Here the $(n + 3)/2$ edges in the cycle receive 1 and $(n - 3) / 2$ edges receive 0; the edges $v_1'v_2$ and $v_1'v_n$ contribute 0.

Therefore $e_f(1) = (n + 3)/2$ and $e_f(0) = (n - 3)/2 + 2 = (n + 1) / 2$

Hence in the above two cases, $|e_f(0) - e_f(1)| \leq 1$

Thus duplication of an arbitrary vertex from C_n forms a divisor cordial graph.

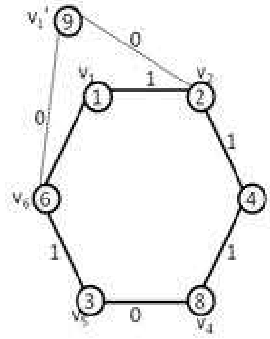

Figure 6. Duplication of v_1 from C_6

Theorem 4.5. The graph obtained by duplication of arbitrary edge in C_n admits divisor cordial labeling.

Proof: Let $v_1, v_2, v_3 \ldots v_n$ be the vertices of the cycle C_n.

Let G be the graph obtained by duplicating an arbitrary edge of C_n.. Without loss of generality, let this edge be $e_1 = v_1v_2$ and newly added edge be $e_1' = v_1'v_2$. Then $E(G) = \{E(C_n) ; e_1', e', e''\}$ where $e' = v_2'v_3$ and $e'' = v_nv_1'$ and $|V(G)| = n + 2$ and $|E(G)| = n + 3$.

Case 1. n is even

We label $f(v_1) = 1$; $f(v_1') = n – 1$; $f(v'') = n + 1$

The v_i vertices are labeled as in (2). Here $(n + 2)/2$ edges receive label 1 and $(n - 2)/2$ edges receive label 0. The vertices v_1' and v_2' have consecutive odd labels and they do not divides, so that the edges $v_1' v_2'$ contributes 0.

The vertex v_3 has label 4 and 4 does not divide an odd number, so that the edge $v_2'v_3$ contribute 0. Similarly v_nv_1' contributes 0. Therefore, $e_f(0) = (n+2)/2$ and $e_f(1) = (n-2) / 2 + 3 = (n + 4) /2$. Hence $| e_f(0) – e_f(1) | \leq 1$.

Case 2. n is odd

Label the vertices v_1' and v_2' as follows: $f(v_1') = n$ and $f(v_2') = n + 2$. Since the vertices in the cycle are labeled as in (2), $(n + 3)/2$ receive label 1, and $(n - 3) /2$ edges receive label 0. As said in the above case, edges v_nv_1', $v_1'v_2'$ and $v_2'v_3$ receive 0. Therefore $e_f(1) = (n + 3)/2$ and $e_f(0) = (n-3)/2 + 3 = (n + 3)/2$. Hence $|e_f(0) – e_f(1)| = 0$

Thus the graph obtained by duplication of arbitrary edge in C_n admits divisor cordial labeling.

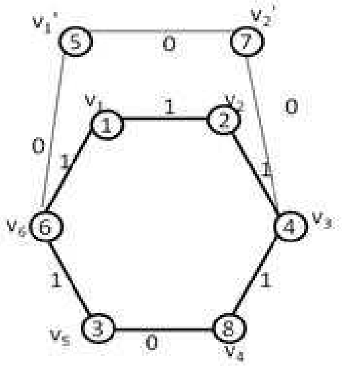

Figure 7. Duplication of v_1v_2 from C_6

Definition 5.1. [5] *Bistar* is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$.

Theorem 5.2. $B_{n,n}$ is a divisor cordial graph.
Proof: Let $B_{n,n}$ be a bistar with vertex set $V(G) = \{u, v\ u_i, v_i, ; 1 \leq i \leq n\}$ where u_i, v_i pendant vertices are and u, v are the apex vertices. Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1$.

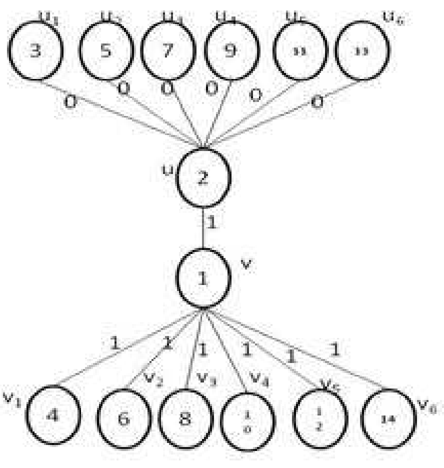

Figure 8. $B_{6,6}$ is divisor cordial

We define f as,

$f(u) = 2$
$f(v) = 1$
$f(v_i) = 2i + 2 ; 1 \leq i \leq n$
$f(u_j) = 2j + 1 ; 1 \leq j \leq n$

The vertex v is labeled as 1. Trivially the (n + 1) edges adjacent to v receive 1. Next, u has been given label 2 and it is adjacent to n number of u_i's which has odd integer label, they does not divide each other and so uv_i's receive 0.
Therefore, $e_f(1) = n + 1$ and $e_f(0) = n$. Hence $| e_f(0) – e_f(1) | = 1$
Thus $B_{n,n}$ is a divisor cordial graph.

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