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Sum Distance in Fuzzy Graphs

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Abstract. In this paper the idea of sum distance which is a metric, in a fuzzy graph is introduced. The concepts of eccentricity, radius, diameter, center and self centered fuzzy graphs are studied using this metric. Some properties of eccentric nodes, peripheral nodes and central nodes are obtained. A characterization of self centered complete fuzzy graph is obtained and conditions under which a fuzzy cycle is self centered are established. We have developed algorithms to obtain the sum distance matrix, eccentricities of the nodes, diameter and radius of a fuzzy graph.

Keywords: Fuzzy graph, Sum Distance, Fuzzy cycle, Eccentricity, Central Nodes.

AMS Mathematics Subject Classification (2010) : 05C72, 05C12, 05C38, 05C40, 90C35.

1. Introduction

A mathematical frame work to describe the phenomena of uncertainty in real life situation has been suggested by Zadeh in 1965 [29]. The advantage of replacing the classical sets by Zadeh's fuzzy sets is that it gives more accuracy and precision in theory and more efficiency and system compatibility in applications. The theory of fuzzy graphs was independently developed by Rosenfeld [19], Yeh and Bang [28] in 1975. Abdul Jabbar *et al.* [1] introduced the concept of fuzzy planar graph. Interval valued fuzzy planar graphs and interval valued fuzzy dual graph are defined by Tarasankar *et al.* in [26]. Some properties of Interval valued fuzzy planar graphs and interval valued fuzzy dual graph are also studied by the authors. Recently, Akram *et al.* introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs [2,5,6,7,8] . Further he has defined length, distance, eccentricity, radius and diameter of a bipolar fuzzy graph and has introduced the concept of self centered bipolar fuzzy graphs[3]. The author has also introduced the concept of an antipodal intuitionistic fuzzy graph and self median intuitionistic fuzzy graph of the given intuitionistic fuzzy graph [4]. To model ecological problems, in 1968 Cohen [14] introduced the notion of competition graphs. Fuzzy competition graph was introduced by Samanta and Pal [20]. Two generalizations of fuzzy competition graph as fuzzy *k*-competition graphs and *p*-competition fuzzy graphs are also defined by the same authors. In [21], Samanta *et al.* define another generalization of fuzzy competition graph, called *m*-step competition graph. Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. He introduced the concept of μ-distance in fuzzy graphs and based on this μ- distance Bhattacharya [9] has introduced the concepts of eccentricity and center in fuzzy graphs and the properties of this metric are further studied by Sunitha and Vijyakumar [25]. Bhutani and Rosenfeld have introduced the concepts of strong arcs [11], fuzzy end nodes [13] and geodesics in fuzzy graphs [12]. Further studies based on the g-distance are carried out by Sameena and Sunitha in [22] and [23]. The concepts of g-peripheral nodes, g-boundary nodes and g-interior nodes based on g-distance ware introduced by Linda and Sunitha [16]. In this paper we introduce the idea of sum distance in fuzzy graphs.

Section 2 contains preliminaries and in section 3, sum distance in fuzzy graphs is defined and proved that it is a metric. Based on this metric, eccentricity, radius, diameter, center in fuzzy graphs are defined. Necessary conditions for a fuzzy graph to be self centered are obtained in this section. By an example it is shown that a unique eccentric node fuzzy graph with each node eccentric need not be self centered. Sufficient conditions for a fuzzy cycle to be self centered is given in section 4. A necessary and sufficient condition for a complete fuzzy graph to be self centered is obtained in section 5. In section 6 we have the embedding theorem i.e. construction of fuzzy graph G from a given fuzzy graph H such that $<C(G)> \approx H$. In section 7, it is shown by examples that the center of a fuzzy tree need not be a fuzzy tree and that there are self centered fuzzy trees. We have given three algorithms in section 8. First algorithm is to generate an adjacency matrix, second algorithm is to find sum distance from adjacency matrix and the third algorithm is to find out the eccentric nodes, diameter and radius of a fuzzy graph G from the distance matrix.

2. Preliminaries

A fuzzy graph(f-graph) [18] is a triplet $G : (V,\sigma,\mu)$ where V the vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)\ \forall u,v \in V$. We assume that V is finite and non empty, μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also we denote the underlying crisp graph [15] by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \text{ x } V : \mu(u, v) > 0\}$. Here we assume $\sigma^* = V$. A fuzzy graph $H : (V,\tau,\upsilon)$ is called a partial fuzzy subgraph of $G : (V,\sigma,\mu)$ if $\tau(u) \leq \sigma(u)\ \forall \text{u} \in \tau^*$ and $\upsilon(u,v) \leq \mu(u,v)\ \forall (u,v) \in \upsilon^*$. In particular we call $H : (V,\tau,\upsilon)$ a fuzzy subgraph of $G : (V,\sigma,\mu)$ if $\tau(u) = \sigma(u)\ \forall \text{u} \in \tau^*$ and $\upsilon(u,v) = \mu(u,v)\ \forall (u,v) \in \upsilon^*$ and if in addition $\tau^* = \sigma^*$, then H is called a spanning fuzzy subgraph of G. A weakest arc of $G : (V, \sigma,\mu)$ is an arc with least membership value. A path P of length n is a sequence of distinct nodes u_0, u_1, ... , u_n such that $\mu(u_{i-1}, u_i) > 0$, $i = 1,2,3,\ldots,n$ and the degree of membership of a weakest arc in the path is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and a cycle P is called a fuzzy cycle (f-cycle) if it contains more than one weakest arc. A fuzzy graph $G : (V,\sigma,\mu)$is said to be complete if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$, $\forall u,v \in \sigma^*$.
Rosenfeld [19] has defined μ-length of any $u-v$ path P as the sum of reciprocals of arc weights in P and distance between u and v called the μ-distance denoted by $d_\mu(u, v)$,

as the smallest μ-length of P. In a fuzzy graph $G : (V,\sigma,\mu)$, $d_\mu(u, v)$ is a metric on V $\forall$ $u,v \in V$. The strength of connectedness between two nodes u and v is defined as the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u,v)$. A $u - v$ path P is called a strongest $u-v$ path if its strength equals $\mathrm{CONN}_G(u, v)$. A fuzzy graph $G : (V, \sigma,\mu)$ is connected if for every u,v in σ^*, $CONN_G(u, v) > 0$. Throughout this, we assume that G is connected. An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted and a $u - v$ path is called a strong path if it contains only strong arcs [11]. A strong path P from u to v is a $u - v$ geodesic if there is no shorter strong path from u to v and the length of a $u - v$ geodesic is the geodesic distance from u to v denoted by $d_g(u, v)$ [12]. The geodesic eccentricity and geodesic center of a fuzzy graph G is also discussed in [12]. Consider the fuzzy graphs $G_1 : (V_1, \sigma_1,\mu_1)$ and $G_2 : (V_2,\sigma_2,\mu_2)$ with $\sigma_1^* = V_1$ and $\sigma_2^* = V_2$. An isomorphism [10] between two fuzzy graphs G_1 and G_2 is a bijective map $h : V_1 \rightarrow V_2$ that satisfies $\sigma_1(u) = \sigma_2(h(u))$ $\forall$ $u \in V_1$ and $\mu_1(u, v) = \mu_2(h(u), h(v))$ $\forall$ $u, v \in V_1$ and is denoted by $G_1 \approx G_2$.

An arc (u,v) is a fuzzy bridge(f-bridge) of G if deletion of (u,v) reduces the strength of connectedness between some pair of nodes [19]. Equivalently, (u,v) is a fuzzy bridge if and only if there exist x,y such that (u,v) is an arc on every strongest $x-y$ path. A node is a fuzzy cutnode (f-cutnode) of G if removal of it reduces the strength of connectedness between some other pair of nodes [19]. Equivalently, w is a fuzzy cutnode if and only if there exist u,v distinct from w such that w is on every strongest $u - v$ path. A connected fuzzy graph $G : (V ,\sigma,\mu)$ is a fuzzy tree (f-tree) if it has a spanning fuzzy subgraph $F : (V ,\sigma,\upsilon)$, which is a tree, where for all arcs (u, v) not in F there exists a path from u to v in F whose strength is more than $\mu(u, v)$. Thus for all arcs (u, v) which are not in F, $\mu(u, v) < CONN_F (u, v)$. Depending on the $CONN_G(u, v)$ of an arc (u, v) in a fuzzy graph G, strong arcs are further classified as $\alpha-$strong & $\beta-$strong and the remaining arcs are termed as $\delta-$arcs [24] as follows. Note that $G - (u, v)$ denotes the fuzzy subgraph of G obtained by deleting the arc (u, v) from G.

Definition 2.1. An arc (u, v) in G is called $\alpha-$strong if $\mu(u, v) > CONN_{G-(u,v)}(u, v)$.

Definition 2.2. An arc (u, v) in G is called $\beta-$strong if $\mu(u, v) = CONN_{G-(u,v)}(u, v)$.

Definition 2.3. An arc (u, v) in G is called a $\delta-$arc if $\mu(u, v) < CONN_{G-(u,v)}(u, v)$.

Definition 2.4. A $\delta-$arc (u, v) is called a δ^*- arc if $\mu(u, v) > \mu(x, y)$ where (x, y) is a weakest arc of G.

3. Sum distance in fuzzy graph

In [27], the authors define the distance $d[\sigma(v_i), \sigma(v_j)]$ between two nodes $\sigma(v_i)$ and $\sigma(v_j)$ in a fuzzy graph as the length of the shortest path between them, i.e. $d[\sigma(v_i), \sigma(v_j)] = \mathrm{Min}[\Sigma_{i,j\in\wedge}\, \mu(v_i, v_j)]$. But this definition does not satisfy the triangle inequality (Fig.1).

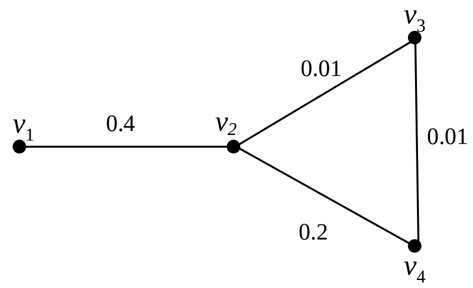


Figure 1: Fuzzy graph

Remark 3.1. In Fig.1 consider the nodes v_1 and v_4 , the two paths joining v_1 and v_4 are $P_1 : v_1 - v_2 - v_4$ and $P_2 : v_1 - v_2 - v_3 - v_4$, of which the shortest path is P_1 . Therefore by the above definition the distance between v_1 and v_4 is the length of P_1. i.e. $d(v_1, v_4) = 0.6$. Proceeding similarly we have $d(v_1, v_3) = 0.41$, $d(v_3, v_4) = 0.01$ and $d(v_1, v_3) + d(v_3, v_4) = 0.42$. Then $d(v_1, v_4) > d(v_1, v_3) + d(v_3, v_4)$, the triangular inequality is not satisfied. Hence the distance d given in the above definition is not a metric. We modify this definition of distance in [27] so that it is a metric.

Definition 3.2. Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph. For any path $P : u_0 - u_1 - u_2 - u_3 - \ldots\ldots - u_n$, length of P is defined as the sum of the weights of the arcs in P i.e. $L(P) = \sum_{i=1}^{n} \mu(u_{i-1}, u_i)$. If $n = 0$, define $L(P) = 0$ and for $n \geq 1$, $L(P) > 0$. For any two nodes u,v in G, let $\mathbf{P} = \{ P_i : P_i$ is a $u - v$ path, $i = 1,2,3,\ldots\}$. The sum distance between u and v is defined as $d_S(u,v) = \text{Min}\ \{L(P_i) : P_i \in \mathbf{P}, i = 1,2,3,\ldots\}$.

Remark 3.3. If $\mu(u,v) = 1\ \forall\ (u,v) \in \mu^*$ then $d_S(u,v)$ is the length of the shortest path as in crisp graph.

Theorem 3.4. *In a fuzzy graph* $G : (V, \sigma, \mu)$, $d_S : V \times V \to [0,1]$ *is a metric on* V. *i.e.* $\forall u,v,w \in V$

(1) $d_S(u,v) \geq 0\ \forall u,v \in V$
(2) $d_S(u,v) = 0$ *if and only if* $u = v$
(3) $d_S(u,v) = d_S(v,u)$
(4) $d_S(u,w) \leq d_S(u,v) + d_S(v,w)$

Proof: (1) and (2) follows from the definition. Next, since reversal of a path from u to v is a path from v to u and vice versa, $d_S(u,v) = d_S(v,u)$. Let P_1 be a $u-v$ path such that $d_S(u,v) = L(P_1)$ and P_2 be a $v - w$ path such that $d_S(v,w) = L(P_2)$. The path P_1 followed by P_2 is a $u-w$ walk and since every walk contains one path, there exists a $u - w$ path in G whose length is at most $d_S(u,v) + d_S(v,w)$.Therefore, $d_S(u,w) \leq d_S(u,v) + d_S(v,w)$.

Definition 3.5. Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph and let u be a node of G. The eccentricity $e(u)$ of u is the sum distance to a node farthest from u. Thus $e(u) =$

$\max\{d_S(u,v) : v \in V\}$. For a node u, each node at sum distance $e(u)$ from u is an eccentric node for u denoted by u^*. G is a unique eccentric node (u.e.n) fuzzy graph if each node in G has a unique eccentric node. The radius $r(G)$ is the minimum eccentricity of the nodes, whereas the diameter $d(G)$ is the maximum eccentricity. A node u is a central node if $e(u) = r(G)$, and $C(G)$ is the set of all central nodes. The fuzzy subgraph induced by $C(G)$ denoted by $<C(G)> = H : (V, \tau, \upsilon)$ is called the center of G. A connected fuzzy graph G is self centered if each node is a central node i.e. $G \approx H$. A node u is a peripheral node if $e(u) = d(G)$.

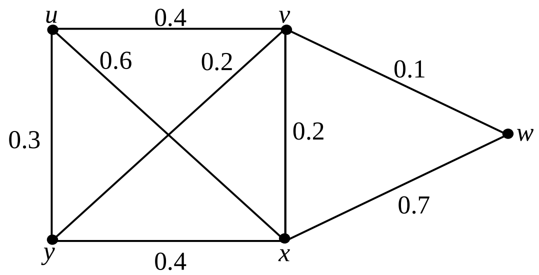


Figure 2: Eccentric nodes, Central nodes, Peripheral nodes

Example 3.6. In Figure 2, $d_S(u,v) = 0.4$, $d_S(u,w) = 0.5$, $d_S(u,x) = 0.6$, $d_S(u,y) = 0.3$, $d_S(v, w) = 0.1$, $d_S(v, x) = 0.2$, $d_S(v, y) = 0.2$, $d_S(w, x) = 0.3$, $d_S(w, y) = 0.3$, $d_S(x, y) = 0.4$. Therefore $e(u) = 0.6$, $u^* = x$, $e(v) = 0.4$, $v^* = u$, $e(w) = 0.5$, $w^* = u$, $e(x) = 0.6$, $x^* = u$, $e(y) = 0.4$, $y^* = x$. The central nodes are v and y. The peripheral nodes are u and x. Here $r(G) = 0.4$ and $d(G) = 0.6$. Note that the f-graph in figure 2 is a u.e.n. f-graph.

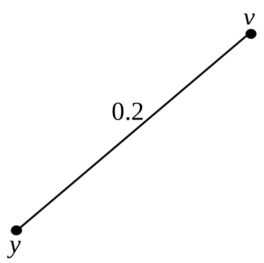


Figure 3: Center of fuzzy graph in figure 2

Theorem 3.7. *For any connected fuzzy graph $G : (V, \sigma, \mu)$, the radius and diameter satisfy $r(G) \le d(G) \le 2r(G)$.*

Proof: $r(G) \le d(G)$ follows from the definition of radius and diameter. Let w be a central node of G. Therefore $e(w) = r(G)$. Let u and v be two peripheral nodes of G. Therefore $e(u) = e(v) = d(G)$.
By triangle inequality $d_S(u,v) \le d_S(u,w) + d_S(w,v)$
i.e. $d(G) \le r(G) + r(G)$. $d(G) \le 2r(G)$. Therefore $r(G) \le d(G) \le 2r(G)$.

Theorem 3.8. *For every two adjacent nodes u and v in a connected fuzzy graph* $G : (V, \sigma, \mu)$, $|e(u) - e(v)| \leq 1$.

Proof: Assume without loss of generality $e(u) \geq e(v)$. Let x be a node farthest from u. i.e. $e(u) = d_S(u,x) \leq d_S(u,v) + d_S(v,x)$, by triangle inequality. Therefore $e(u) \leq d_S(u,v) + e(v)$, since $e(v) \geq d_S(v, x)$. Since u and v are adjacent nodes we have $d_S(u, v) \leq 1$. Therefore $e(u) \leq 1 + e(v) \Rightarrow 0 \leq e(u) - e(v) \leq 1$. Therefore, $|e(u) - e(v)| \leq 1$.

The above theorem can be generalized as follows.

Theorem 3.9. *For every two nodes u and v in a connected fuzzy graph* $G : (V, \sigma, \mu)$, $|e(u) - e(v)| \leq d_S(u,v)$.

Proof: Assume without loss of generality $e(u) \geq e(v)$. Let x be a node farthest from u. i.e. $e(u) = d_S(u, x) \leq d_S(u, v) + d_S(v, x)$, by triangle inequality. Therefore $e(u) \leq d_S(u, v) + e(v)$, since $e(v) \geq d_S(v,x)$. i.e. $0 \leq e(u) - e(v) \leq d_S(u, v)$. $\therefore |e(u) - e(v)| \leq d_S(u,v)$.

Theorem 3.10. *Let u and v be adjacent nodes in a connected fuzzy graph* $G : (V, \sigma, \mu)$, *then* $|d_S(u, x) - d_S(v, x)| \leq 1$ *for every node x of G.*

Proof: Let u and v be adjacent nodes in G and let x be any node of G. Assume $d_S(u, x) \geq d_S(v, x)$. Then by triangle inequality we have $d_S(u, x) \leq d_S(u, v) + d_S(v, x)$. Since u and v are adjacent nodes $d_S(u, x) \leq 1 + d_S(v, x) \Rightarrow 0 \leq d_S(u, x) - d_S(v, x) \leq 1$. $\therefore |d_S(u, x) - d_S(v, x)| \leq 1$.

Remark 3.11. For any two real numbers a, b such that $0 < a \leq b \leq 2a$, there exist a fuzzy graph G such that $r(G) = a$ and $d(G) = b$.

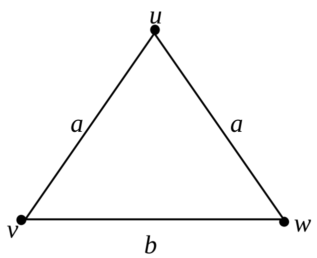


Figure 4: Fuzzy graph

In figure 4, $d_S(u, v) = a$, $d_S(u, w) = a$ and $d_S(v, w) = b$. Then $e(u) = a$, $e(v) = b$ and $e(w) = b$. Therefore $r(G) = a$ and $d(G) = b$.

Theorem 3.12. *If* $G : (V, \sigma, \mu)$ *is a self centered fuzzy graph, then each node of G is eccentric.*

Proof: Assume G is self centered and let u be any node of G. Let v be an eccentric node of u i.e. $u^* = v$. Then $e(u) = d_S(u, v)$. Since G is self centered we have $e(v) = e(u)$. Therefore $e(u) = d_S(u, v) = e(v)$, which shows u is an eccentric node of v i.e. $v^* = u$. Hence the proof.

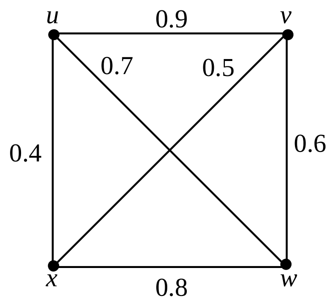


Figure 5: Fuzzy graph which is not self centered

Remark 3.13. The condition in Theorem 3.12 is not sufficient. In figure 5, each node is eccentric but G is not self centered. Here $r(G) = 0.8$ and $d(G) = 0.9$ and w and x are the central nodes

Theorem 3.14. *If $G : (V, \sigma, \mu)$ is a self centered fuzzy graph, then for every pair of nodes $u,v \in G$, $u \in V^*$ implies $v \in U^*$, where U^* is the set of all eccentric nodes of u and V^* is the set of all eccentric nodes of v.*

Proof: Assume G is self centered and let u,v be any two nodes of G. Let u be an eccentric node of v. i.e. $d_S(v, u) = e(v)$, so we have $u \in V^*$. Now required to prove that $v \in U^*$. Since G is self centered we have $e(v) = e(u)$. Also we have $d_S(v, u) = d_S(u, v) = e(v)$. Therefore $e(u) = d_S(u, v)$ which shows v is an eccentric node of u i.e. $v \in U^*$. Hence the proof.

Remark 3.15. The condition in Theorem 3.14 is not sufficient. In figure 5 each node is eccentric and we have $u^* = v$, $v^* = u$ and $w^* = x$, $x^* = w$ but G is not self centered.

Remark 3.16. A unique eccentric node f-graph with each node eccentric need not be self centered. In figure 5 each node is eccentric and each node has a unique eccentric node but G is not self centered.

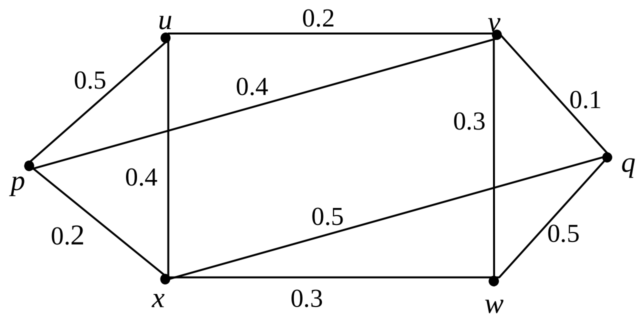
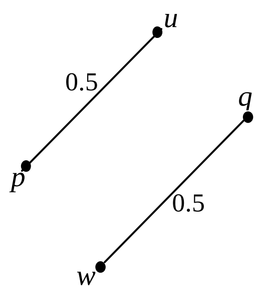


Figure 6: Fuzzy graph $G : (V, \sigma, \mu)$



<C(G)>

Remark 3.17. The center of a connected f-graph need not be connected as shown in Figure 6 .

Theorem 3.18. *In a fuzzy graph $G:(V,\sigma,\mu)$ all peripheral nodes are eccentric nodes.*

Proof: Let u be a peripheral node of G. Therefore $e(u) = d(G)$ and there exist at least one node v in G such that $e(u) = d_S(u,v) = d(G)$. Therefore $v^* = u$, i.e. u is an eccentric node of v.

Remark 3.19. The converse of Theorem 3.18 is not true. In figure 5, *u, v, w, x* are eccentric nodes but only *u* and *v* are peripheral nodes.

Remark 3.20. There are fuzzy graphs with peripheral nodes as fuzzy cut nodes. In figure 2 , nodes *u* and *x* are peripheral nodes as well as fuzzy cut nodes. Note that removal of the node *u* reduces strength of connectedness between the nodes *v* & *x* and removal of the node *x* reduces strength of connectedness between the nodes *u* & *y* .

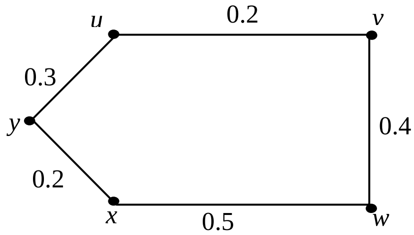


Figure 7: Fuzzy cycle which is not self centered

Remark 3.21. A fuzzy cycle need not be self centered. In Figure 7, $r(G) = 0.6$ and $d(G) = 0.7$ and the central node is *u*.

4. Self centered fuzzy cycle

Using the concept of μ−eccentric nodes, in [25] Sunitha and Vijayakumar has proved the sufficient conditions for a fuzzy graph G such that G^* is a cycle to be self centered. In this section, sufficient conditions for a cycle to be self centered based on sum distance is discussed.

Theorem 4.1. *Let $G:(V,\sigma,\mu)$ be a fuzzy graph with n nodes such that $G^* \approx C_n$, cycle on n nodes with arcs $e_i = (u_i, u_{i+1})$ $i = 1,2,\cdots,n-1$ and $e_n = (u_n, u_1)$. Let $0 < t < s \le 1$. Then G is self centered if*

i) $\mu(e_i) = t$ for $i = 1,3,5,\cdots,n-1$, $\mu(e_i) = s$ for $i = 2,4,6,\cdots,n-2$ and $\mu(e_n) = s$ when n is even.

ii) $\mu(e_i) = s$ for $i = 1,3,5,\cdots,n-2$, $\mu(e_i) = t$ for $i = 2,4,6,\cdots,n-1$ and $\mu(e_n) = s$ when n is odd and $n = 4k-1$, where $k = 1,2,3,\cdots$.

iii) $\mu(e_i) = t$ for $i = 1,3,5,\cdots,n-2$, $\mu(e_i) = s$ for $i = 2,4,6,\cdots,n-1$ and $\mu(e_n) = t$ when n is odd and $n = 4k+1$, where $k = 1,2,3,\cdots$.

Also,

$$r(G) = \begin{cases} k(t+s),\ n = 4k \text{ or } n = 4k+1,\ k = 1,2,3,\cdots \\ k(t+s)-t,\ n = 4k-1,\ k = 1,2,3,\cdots \\ k(t+s)+t,\ n = 4k+2,\ k = 1,2,3,\cdots \end{cases}$$

Illustration 1. Take $t = 0.3$ and $s = 0.4$

Case 1. n is even and $n = 4k$ where $k = 1,2$, $r(C_4) = 0.7$ and $r(C_8) = 1.4$.

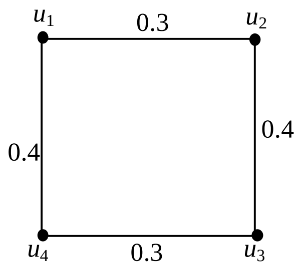


Figure 8: C_4

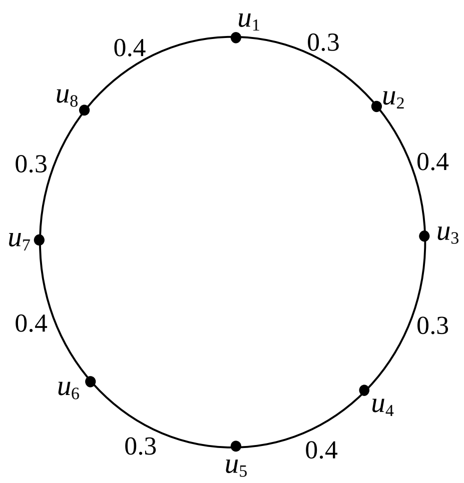


Figure 9: C_8

Case 2. n is even and $n = 4k + 2$ where $k = 1,2$, $r(C_6) = 1.0$ and $r(C_{10}) = 1.7$.

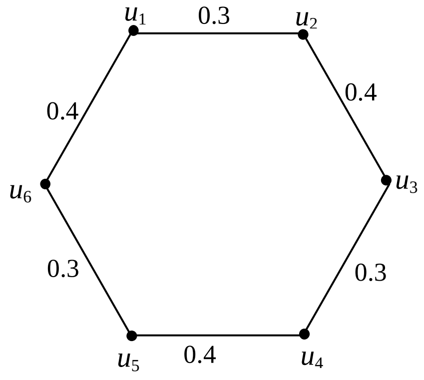


Figure 10: C_6

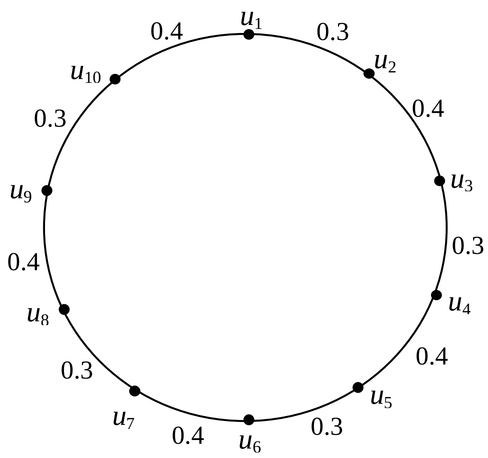


Figure 11: C_{10}

Case 3. n is odd and $n = 4k - 1$ where $k = 1,2$, $r(C_3) = 0.4$ and $r(C_7) = 1.1$.

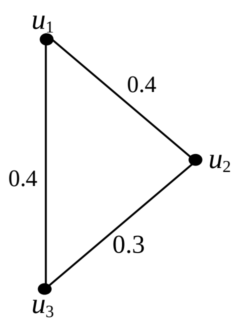


Figure 12: C_3

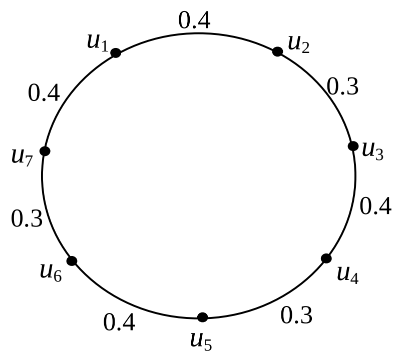


Figure 13: C_7

Case 4. n is odd and $n = 4k + 1$ where $k = 1,2$, $r(C_5) = 0.7$ and $r(C_9) = 1.4$.

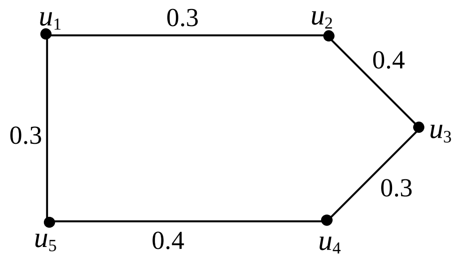


Figure 14: C_5

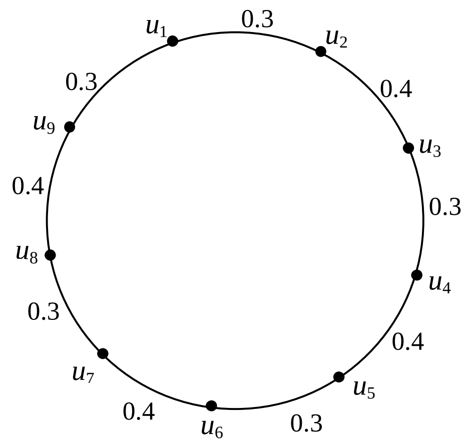


Figure 15: C_9

5. Sum distance in complete fuzzy graph

In [17] Mini and Sunitha proved that any $u - v$ path P in a CFG is a strongest path if and only if either u or v is a weakest node in the path. In this section we first prove a necessary and sufficient condition for all paths in a CFG to be strongest and then a necessary and sufficient condition for a CFG to be self centered.

Remark 5.1. A complete fuzzy graph need not be self centered. In figure 16, $r(G) = 0.3$ and $d(G) = 0.5$ and the central node is u.

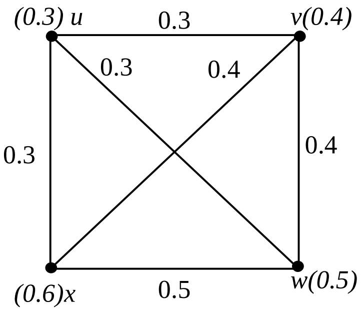


Figure 16: Complete fuzzy graph which is not self centered

Theorem 5.2. *Let $G : (V, \sigma, \mu)$ be a CFG with $\sigma^* = \{u_1, u_2, u_3, ...u_n\}$ such that $\sigma(u_1) \leq \sigma(u_2) \leq \sigma(u_3) \leq \cdots\cdot \leq \sigma(u_n)$. Then the sum distance between any two nodes u_i, uj in G is either $\mu(u_i, u_j)$ or $2\sigma(u_1)$.*

Proof: Let u_i, u_j be any two nodes in G. We have $d_S(u_i, u_j) = \min\{\mu(u_i, u_j), \mu(u_i, u_k) + \mu(u_k, u_j)\}$. Since G is CFG we have $\mu(u_i, u_k) = \sigma(u_i) \wedge \sigma(u_k)$. Also since $\sigma(u_1) \leq \sigma(u_i)$ for $i = 2,3,\cdots,n$, when $k = 1$, $\mu(u_i, u_1) = \sigma(u_1)$ and $\mu(u_1, u_j) = \sigma(u_1)$. Therefore $d_S(u_i, u_j) = \min\{\mu(u_i, u_j), 2\sigma(u_1)\}$.

Theorem 5.3. *Let $G : (V, \sigma, \mu)$ be a CFG on n nodes, $n \geq 3$. All paths in G are strongest paths if and only if there is at most one node w in G having different node strength and $\sigma(w) > \sigma(u_i)$ $i = 1,2,3,\cdots,n-1$.*

Proof: Let $G : (V, \sigma, \mu)$ be a CFG. Assume all paths in G are strongest paths. Suppose $\exists$ two nodes v, w in G having different strength. i.e. $\sigma(v) \neq \sigma(u_i)$ and $\sigma(w) \neq \sigma(u_i)$, $i = 1,2,3,\cdots,n-2$.

Case 1 : $\sigma(u_i) < \sigma(w)$ and $\sigma(u_i) < \sigma(v)$, $i = 1,2,3,\cdots,n-2$.

Let $P : w - u_1 - u_2 - .... - u_k - v$, $k \leq n-2$, be a $w - v$ path. Then P is not a strongest $w - v$ path since neither w nor v is a weakest node in P [17], contradiction.

Case 2 : $\sigma(u_i) > \sigma(w)$ and $\sigma(u_i) > \sigma(v)$, $i = 1,2,3,\cdots,n-2$.

Let P be any $u_i - u_j$ path, $i, j = 1,2,3,\cdots,n-2$, $i \neq j$ with either v or w as an internal node. Then P is not a strongest $u_i - u_j$ path since neither u_i nor u_j is a weakest node in P [17], contradiction .

Case 3 : $\sigma(u_i) < \sigma(w)$ and $\sigma(u_i) > \sigma(v)$, $i = 1,2,3,\cdots,n-2$.

Let P be any $u_i - w$ path, $i = 1,2,3,\cdots,n-2$, with v as an internal node. Then P is not a strongest $u_i - w$ path since neither u_i nor w is a weakest node in P [17], contradiction. Hence there exist at most one node w in G having different node strength. Next to prove $\sigma(w) > \sigma(u_i)$ $i = 1,2,3,\cdots,n-1$. Suppose not let, $\sigma(w) < \sigma(u_i)$ $i = 1,2,3,\cdots,n-1$.Then by case 2, we arrive at a contradiction. Hence $\sigma(w) > \sigma(u_i)$ $i = 1,2,3,\cdots,n-1$.

Conversely assume that there is at most one node w in G having different node strength and $\sigma(w) > \sigma(u_i)$ $i = 1,2,3,\cdots,n-1$. Then any path P, joining any two nodes in G is such that at least one of the end nodes of P is a weakest node in the path P and hence P is a strongest path [17].

Theorem 5.4. *Let $G : (V, \sigma,\mu)$ be a CFG on n nodes, $n \geq 3$. Then G is self centered if and only if all paths in G are strongest paths.*

Proof: Let $G : (V, \sigma,\mu)$ be a CFG . Assume G is self centered. Then by Theorem 3.12, each node of G is eccentric. Also for any two nodes u,v in G, $e(u) = e(v) = r(G) = d(G)$. If possible assume that all paths in G are not strongest paths. Therefore by Theorem 5.3, there exist at least two nodes u, v with different node strength and let w be an arbitrary node in G such that $\sigma(w)$ is least. i.e. we have $\sigma(w) < \sigma(u)$ and $\sigma(w) < \sigma(v)$. Also we have $\mu(u, v) = \sigma(u) \wedge \sigma(v) > \sigma(w)$ and $d_S(u, v) = \min\{ \mu(u, v) , 2\sigma(w) \}$ by Theorem 5.2. Therefore $d_S(u,v) > \sigma(w)$. Also we have $e(u) = \max\{d_S(u,v) : v \in V\}$.

Therefore $e(u) > \sigma(w)$ (1)

Now, for any node u in G we have $\mu(u,w) = \sigma(w)$ and therefore $d_S(u,w) = \sigma(w)$ by Theorem 5.2.

Thus $e(w) = \max\{d_S(w,u) : u \in V\} = \sigma(w)$ (2)

From (1) and (2) $e(u) > e(w)$, which contradicts our assumption that G is self centered. Hence all paths in G are strongest paths.

Conversely assume all paths in G are strongest paths. Since all paths in G are strongest paths, there is at most one node in G having different strength and the strength of such a node is greater than the strength of all other nodes in G by Theorem 5.3. Hence all arcs in G have same strength. Also $d_S(u,v) = \mu(u,v)$ $\forall u,v$ by Theorem 5.2. Hence for any two node u,v in G, $e(u) = e(v)$. Therefore G is self centered.

6. Embedding Theorem

In this section, we shall consider the construction of a fuzzy graph G from a fuzzy graph H such that $< C(G) > \approx H$.

Theorem 6.1. *$H : (V, \sigma', \mu')$ be a fuzzy graph. Then there exists a connected fuzzy graph $G : (V, \sigma,\mu)$ such that $< C(G) > \approx H$.*

Proof: Let $0 < c = \wedge\sigma'(u)$. Construct a fuzzy graph $G : (V, \sigma,\mu)$ from H as follows. Take four new nodes u_1,u_2,v_1,v_2 and put $\sigma^* = \sigma'^* \cup \{ u_1, u_2, v_1, v_2 \}$ where $\sigma = \sigma'$ for all nodes w in H, $\mu = \mu'$ for all arcs (u, v) in H. Let $\sigma(u_i) = \sigma(v_i) = t$ $(t \leq c), i = 1,2$; $\mu(u_1, u_2) = \mu(v_1, v_2) = t$ and $\mu(u_2, w) = \mu(v_1, w) = t$ $\forall$ $w \in H$. Then clearly $G : (V, \sigma,\mu)$ is a fuzzy graph and $e(w) = 2t$ $\forall$ $w \in H$ and $e(u_1) = e(v_1) = 3t$ and $e(u_2) = e(v_2) = 4t$. Thus $< C(G) > \approx H$ and $r(G) = 2t$ and $d(G) = 4t$.

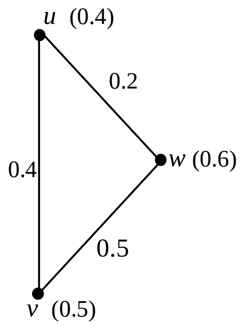


Figure 17: *H : (V,σ′,μ′)*

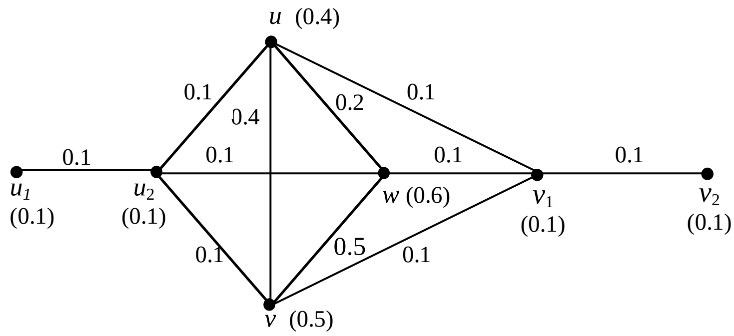


Figure 18: *G : (V,σ,μ)* where $< C(G) > \approx H$.

7. Center of a fuzzy tree

A study of $\mu-$ distance in a fuzzy graph $G : (V, \sigma,\mu)$, which is a fuzzy tree is carried out in [25]. In this section a similar study is carried out on fuzzy trees using sum distance.

Remark 7.1. It is well known that center of a tree is either K_1 or K_2. But, for a fuzzy tree it need not be so as in figure 19.

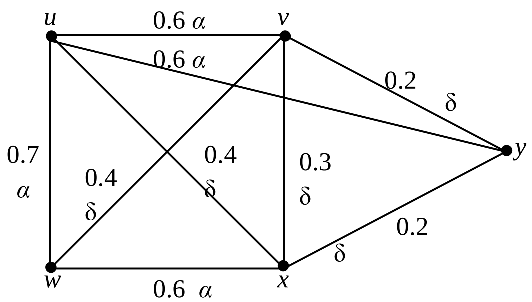
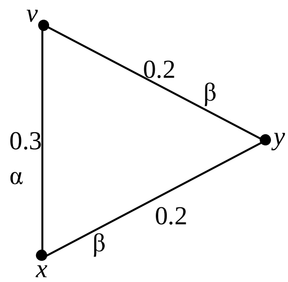


Figure 19: Fuzzy Tree *G : (V,σ,μ)*



< C(G) >

Remark 7.2. In fact, there are self centered fuzzy trees. In figure 20, $G : (V, \sigma,\mu)$ is an f-tree which is self centered with $e(u_i) = 0.5$, i =0,1,2,3.

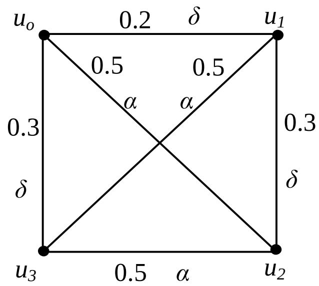


Figure 20: Self centered fuzzy tree

Remark 7.3. The center of a fuzzy tree need not be a fuzzy tree (Figure 19). Note that an f-graph $G:(V,\sigma,\mu)$ is an f-tree if and only if it has no $\beta-$strong arcs [24].

8. Algorithm

In this section we present three algorithms. Let $G:(V,\sigma,\mu)$ be a connected fuzzy graph with n nodes $u_0, u_1, u_2, \cdots\cdots, u_{n-1}$. Let $\mu(u_i, u_j)$ be the arc strength. First algorithm is to generate the adjacency matrix $A=(a_{ij})$ where

$$\boldsymbol{a_{ij}} = \begin{cases} \mu(u_i,u_j)\ , & i \neq \boldsymbol{j} \\ 0\ , & i = \boldsymbol{j} \end{cases}$$

Note that in the adjacency matrix, any non zero element other than the diagonal element is replaced by n, the number of nodes in G, to apply the second algorithm.

Second algorithm is to find sum distance matrix from the adjacency matrix.

Third algorithm is to find out the eccentricities of the nodes, diameter and radius of a fuzzy graph G from the distance matrix.

Algorithm 8.1: ADJACENCY(a,n)

Comment: Input : a is an array representing strength of arcs in the f-graph

Comment: Input : n is number of nodes of the f-graph

```
for i ← 0 to n − 1
do
  for j ← 0 to n − 1
  do
    if  j = i
      then a[i,j] ← 0
    else
      if  j > i
      then
        a[i,j] ← arc strength
        if a[i,j] = 0
          then  a[i,j] ←  n
        a[j,i]←a[i,j]
```

Algorithm 8.2: DISTANCE MATRIX(a,n)

Comment: Input : a is an array representing elements of the adjacency matrix

Comment: Input : n is number of nodes of the f-graph

for $k \leftarrow 0$ **to** $n-1$
  do
    for $i \leftarrow 0$ **to** $n-1$
    do
      $mtemp \leftarrow 0$
      for $j \leftarrow 0$ **to** $n-1$
      do
        if $j > i$
        then
          $mtemp \leftarrow a[i,k] + a[k,j]$
          if $mtemp < a[i,j]$
          then
            $a[i,j] \leftarrow mtemp$
            $a[j,i] \leftarrow a[i,j]$

The complexity of this algorithm is $O(n^3)$.

Algorithm 8.3: ECCENTRICITY(a, n)

Comment: Input : a is an array representing elements of the distance matrix
Comment: Input : n is number of nodes of the f-graph

for $i \leftarrow 0$ **to** $n-1$
  do
    $max \leftarrow 0$
    for $j \leftarrow 0$ **to** $n-1$
    do
      if $a[i,j] > max$
      then $max \leftarrow a[i,j]$
    $e[i] \leftarrow max$
    if $i = 0$
    then
      $dia \leftarrow e[i]$
      $rad \leftarrow e[i]$
    else
      if $e[i] > dia$
      then $dia \leftarrow e[i]$
      if $e[i] < rad$
      then $rad \leftarrow e[i]$

The complexity of this algorithm is $O(n^2)$.

9. Conclusion

The idea of sum distance which is a metric, in a fuzzy graph is introduced. The concepts of eccentricity, radius, diameter, center, self centered f-graphs etc. are studied using this metric. A characterization of self centered complete fuzzy graph is obtained and conditions under which a fuzzy cycle is self centered are established. A necessary and sufficient condition for all paths in a CFG with $n \geq 3$ to be strongest paths is obtained. It is shown by an example that center of a fuzzy tree need not be a fuzzy tree and there are self centered fuzzy trees. Also discussed the construction of a fuzzy graph G from a given fuzzy graph H such that $<C(G)> \approx H$. An algorithm is developed to find sum distance matrix of a fuzzy graph.

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